Self-Fulfilling Debt Dilution: Maturity and Multiplicity in Debt Models*

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Abstract

We establish that creditor beliefs regarding future borrowing can be self-fulfilling, leading to multiple equilibria with markedly different debt accumulation patterns. We characterize such indeterminacy in the Eaton-Gersovitz sovereign debt model augmented with long maturity bonds. Two necessary conditions for the multiplicity are: (i) the government is more impatient than foreign creditors, and (ii) there are deadweight losses from default. The multiplicity is dynamic and stems from the self-fulfilling beliefs of how future creditors will price bonds; long maturity bonds are therefore a crucial component of the multiplicity. We introduce a third party with deep pockets to discuss the policy implications of this source of multiplicity and identify the potentially perverse consequences of traditional "lender of last resort" policies.

1 Introduction

The recent sovereign debt crisis in Europe, along with the associated policy responses, underscores the importance of self-fulfilling debt crises. We introduce and analytically solve a tractable version of the canonical Eaton and Gersovitz (1981) sovereign debt model with long duration bonds and study the vulnerability to self-fulfilling debt crises. The Eaton-Gersovitz model, enhanced to incorporate long-term bonds, has become the workhorse paradigm for a large quantitative literature that has successfully explained key empirical features of sovereign default.[1]

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[1] Examples, among many others, are Aguiar and Gopinath (2006), Arellano (2008), Yue (2010), Hatchondo and Martinez (2009), Mendoza and Yue (2012), Chatterjee and Eyigungor (2012), Arellano and Ramanarayanan (2012), and Bianchi, Hatchondo and Martinez (forthcoming). See Aguiar and Amador (2014) for a survey.

However, due to the intractability of the model, it is not known whether and under what circumstances this environment generates self-fulfilling debt crises.[2] This is a major shortcoming, as long-term bonds are the primary source of government financing around the world. Moreover, they play a key role in bringing the quantitative sovereign debt models closer to the data, in large part due to the inherent incentive to dilute bondholders. We establish that the same force generates multiplicity.

Our analysis introduces a tractable version of the Eaton-Gersovitz model for which we solve for equilibrium objects explicitly. We show that as long as the government is relatively impatient and there are deadweight costs to default, there is a parameter configuration and a maturity of debt that supports multiple equilibria.

The multiplicity is dynamic. Creditor expectations of future borrowing and default behavior determine bond prices today. In turn, current and anticipated bond prices affect the government's incentives to borrow. To shed light on this feedback mechanism, we characterize two types of equilibria with markedly different debt dynamics. In a "borrowing" equilibrium, the government issues bonds until it reaches an endogenous debt limit. In a "saving" equilibrium, the government reduces its stock of debt until default no longer occurs with positive probability. The tension at work in both equilibria is the relative impatience of the government and the deadweight costs of default.

The government saves in order to enjoy high prices when it rolls over the remaining debt in the future. However, this incentive is only operable if there is a deadweight loss in default; as prices are actuarially fair in any equilibrium, they do not provide an incentive to save when default is zero sum.[3] Hence, the combination of deadweight costs and the need to roll over maturing debt provides the foundation for the saving equilibrium.

The government's relative impatience provides a countervailing force that supports the borrowing equilibrium. In the borrowing equilibrium, creditors anticipate future borrowing going forward (that is, "debt dilution"), and prices are low regardless of the current level of indebtedness. In this equilibrium, there is no reward for keeping debt low due to creditor beliefs about future debt dynamics. Hence, whether relative impatience or deadweight costs of default are the dominant force in determining debt dynamics depends on creditor beliefs.

Maturity plays a key role in this indeterminacy, which arises only when debt is of intermediate maturity. When maturity is sufficiently long, the saving equilibrium cannot be supported, as the

[2]In a recent contribution, Auclert and Rognlie (2016) show that the Eaton-Gersovitz model *with one-period bonds* features a unique equilibrium, but their arguments do not extend to long-term bonds. See also Aguiar and Amador (2019).

[3]While lenders receive zero in the default state, a deadweight cost implies the government's value is strictly less than that associated with zero debt. Competitive bond markets imply that creditors are compensated in expectation for the full loss, while the government does not reap the same expected gain. This provides the government with an incentive to reduce the probability of default.

amount of debt to be rolled over at high prices is too small to warrant saving. In particular, as the probability of default is reduced, the gain from the reduction in the deadweight costs of default is split between the government, which is issuing new debt at high prices, and holders of non-maturing bonds, who enjoy a capital gain. As the latter component is irrelevant for the government's decision to save, longer maturity bonds eliminate the government's incentive to save.

Conversely, at very short maturities, the government internalizes the gains from reducing the probability of default. In fact, we show that as maturity becomes arbitrarily short, the government's fiscal policy approaches what would be chosen in a constrained efficient contract between the lenders and the government, as in Aguiar, Amador, Hopenhayn and Werning (2018). In this case, the borrowing equilibrium becomes impossible to sustain without a high degree of relative impatience or zero deadweight costs. For intermediate values of maturity, impatience, and deadweight costs, *either equilibrium can be sustained.*

We show that this multiplicity has novel implications for the design of third-party programs to eliminate inefficient equilibria. Common prescriptions motivated by rollover crisis intuition, such as price floors or emergency lending when spreads are high, may have the perverse outcome of eliminating the preferred equilibrium in the Eaton-Gersovitz model. In our framework, a floor on prices does not eliminate the borrowing equilibrium; in fact, it may eliminate the saving equilibrium and select the borrowing equilibrium. The saving equilibrium requires a steep gradient in prices across the domain of debt to incentivize saving (or prevent dilution). A price floor that extends across a wide range of debt levels eliminates this important feature of the saving equilibrium. A more effective policy to prevent borrowing would be to either limit debt explicitly or promise a price floor conditional on remaining within an exogenous bound on debt that is strictly tighter than the equilibrium debt limit. Such a policy would select the saving equilibrium and not require on-equilibrium resources. However, as with the lender of last resort, off-equilibrium credibility is key. The failure of such explicit debt limits in Europe (and traditional conditionality of the IMF) suggests that such credibility is difficult to establish in practice.[4]

The main analysis uses a tractable, continuous time model. Using a slightly modified version of Chatterjee and Eyigungor (2012) (henceforth, CE12), we also confirm that such multiplicity exists in the standard quantitative model. We adopt CE12's framework largely unchanged, save for one modification to the endowment process. Specifically, motivated by the work of Barro and Ursa (2008) and others, we add a rare-disaster state, in which the endowment falls sharply.[5] The modified CE12 model features (at least) two equilibria at the calibrated expected maturity of

[4] Bocola and Dovis (2016) explore the efficacy of a price floor in a quantitative model of the European debt crisis. The policy they consider to rule out rollover crises similarly imposes a price floor combined with a debt limit.

[5] Ayres, Navarro, Nicolini and Teles (2015), Rebelo, Wang and Yang (2019), and Paluszynski (2019) introduce rare disasters in a quantitative sovereign debt model.

20 quarters. There is a "borrowing" equilibrium, in which the ergodic distribution features high debt and recurrent default, which has similar quantitative properties as the calibration reported in CE12. For the same parameterization, there is also a "saving" equilibrium, in which an indebted government saves in order to attain a risk-free price.

By varying the maturity, we successfully compute a saving equilibrium for maturities ranging from 1 to 33 quarters. The borrowing equilibrium can be computed for maturities as short as 9 quarters. Thus there is a significant range of empirically relevant maturities that support multiple equilibria. By varying the default cost in the disaster state, we use the quantitative model to explore the role of deadweight costs in generating multiplicity. For very low default costs, the one-period maturity version converges to a borrowing equilibrium and we were unable to compute a saving equilibrium. However, as we increase default costs, both a borrowing and a saving equilibrium can be supported.

These experiments reveal two lessons for quantitative sovereign debt models. One is that multiplicity is possible in such models for a wide range of empirically relevant maturities, as long as default costs are not too small. The second is that the practice of calibrating nonlinear default costs in order to match debt and default frequencies in the data may naturally lead to environments in which an absorbing Safe Zone is not constrained efficient, and hence a saving equilibrium may be unlikely to exist. However, such low default costs are not directly tied to empirical evidence and this practice may provide an incomplete picture regarding the vulnerability to self-fulfilling dilution.

The recent literature exploring multiplicity has built on two canonical frameworks, namely, the works of Calvo (1988) and Cole and Kehoe (2000). The Calvo multiplicity arises due to the feedback of prices to the budget set. This is easiest to see in a framework in which the government is forced to raise a certain amount of revenue from a bond auction. A low price (or high spread) for bonds forces the government to issue a greater quantity of debt in terms of face value. This raises the debt payments going forward, increasing the incentive to default and therefore supporting the low price at auction. Conversely, a high price requires lower debt payments and thus may also be an equilibrium. Calvo-style multiplicity is studied in dynamic settings by Lorenzoni and Werning (2013) and Ayres et al. (2015).

Lorenzoni and Werning (2013) provide an antecedent to our paper by analyzing the role of long-term bonds in dynamic settings with multiple equilibria. In an environment where the government follows a fiscal rule, they show how Calvo-style multiplicity arises and how longer debt maturity contributes towards uniqueness.[6] Closer to our current environment, they also discuss how multiplicity arises in a model where the government endogenously chooses its expenditures

[6]In their benchmark model, they show that issuing longer maturities shrinks the region of multiplicity and helps select the "good" equilibrium, while the reverse is true in our analysis.

(rather than following a pre-specified rule), but faces constraints in its ability to reduce the deficit when confronted by adverse bond prices. They uncover an equilibrium where the government saves and bond prices are high, as well as other equilibria where the government instead borrows and prices are low (when the debt is high enough). However, as we discuss in detail in Section 8, the mechanism we identify as generating multiplicity is distinct. Lorenzoni and Werning (2013) emphasizes the *limits* to fiscal discretion when bond prices are low. In contrast, we emphasize the *absence* of such limits. The fact that limited commitment to fiscal policy gives rise to multiplicity when the government issues bonds of intermediate maturities is the novel insight of our paper.

The Cole-Kehoe multiplicity is a "static" multiplicity. Specifically, holding future equilibrium behavior constant, the market clearing price for bonds is not determined. A high price for bonds allows the government to roll over its maturing debt. However, a zero price forces the government to repay all maturing bonds out of current endowment, making default optimal.[7] This type of multiplicity has been extended recently by Aguiar, Chatterjee, Cole and Stangebye (2017) and explored quantitatively by Bocola and Dovis (2016). In our framework, the multiplicity is inherently dynamic in that future expectations over future equilibrium behavior are crucial in supporting the alternative equilibria. The Cole-Kehoe multiplicity emphasizes the vulnerability of short-maturity bonds to crises and favors lengthening maturity to avoid self-fulfilling crises. Our analysis shows that such lengthening opens up the economy to both inefficiencies and a new form of multiplicity.

A recent paper, Stangebye (2018), shares our interest in multiplicity in a Eaton-Gersovitz framework. Stangebye computationally constructs a version in which there exists two Markov equilibria. Interestingly, Stangebye emphasizes concavity of the utility function as crucial in supporting multiplicity. The multiplicity we identify, on the other hand, exists whether the government is risk-neutral or has concave utility. In Section 8, we discuss additional equilibria that arise with a lower bound on consumption, which may play the same role as concavity in Stangebye's analysis. Nevertheless, given the common structure, there are many points of overlap in the nature of the multiplicity studied in the two independent papers, and we view our analysis as complementary to Stangebye's.

The rest of the paper is as follows: Section 2 lays out our benchmark analytical model; Section 3 discusses efficient allocations from a benchmark planning problem; Section 4 contains the main analysis of the alternative equilibria; Section 5 discusses the role of maturity in generating multiplicity; Section 6 explores how commonly proposed third-party policies may or may not select

[7] A related point on the possibility of a liquidity crisis in sovereign debt markets had been made by Sachs (1984) in a model with bank lending. Defaulting because of the inability to roll-over maturing debt generates coordination failures on the lenders side. Detragiache (1996) presents a related analysis of how investment can also generate multiple equilibria. In both of these papers, the multiplicity arises even with finite horizons. See also the recent work of Galli (2019).

a particular equilibrium; Section 7 shows how the theoretical insights extend to the richer environments used in quantitative analysis; Section 8 discusses the relationship with other sources of multiplicity; and Section 9 concludes.

2 Environment

We study an infinite-horizon small open economy. Time is continuous and indexed by t. The economy receives a constant flow endowment y. Consumption and savings decisions for the economy are made by a government. The government has access to a non-contingent bond that it trades with atomistic, risk-neutral lenders. The lenders discount at the world risk-free interest rate $R = (1 + r)$. The small open economy assumption implies that R is invariant to the government's borrowing or default decisions. Lenders have sufficient wealth as a group to hold an arbitrary quantity of bonds.

The asset space is restricted to a single type of bond. To incorporate maturity in a tractable manner, we follow Leland (1994), Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012) by considering random maturity bonds. A bond matures with a constant hazard rate δ, at which point a payment of 1 is required. We assume that bonds mature independently such that a deterministic fraction δ of any portfolio of bonds matures each instant. The expected life span of a bond is $1/\delta$; hence, δ is a measure of (inverse) expected maturity. The advantage of this formulation is that all bonds that have yet to mature are identical; in particular, they all have the same expected maturity going forward regardless of when they were issued.

We normalize the coupon of a bond to be the risk-free rate r. That is, a bond pays a flow coupon r each instant through maturity. This implies that a risk-free bond has price 1 in equilibrium, which serves as the upper bound on the price of the sovereign's bond.

If the government misses a coupon or principal payment, it is in default. As in Aguiar et al. (2018), the value of default is a random variable and captures any punishment that can be imposed by creditors, including lost endowment, as well as any utility costs (or benefits) to the government from defaulting. Changes in the value of default represent the source of risk to creditors in our analysis.

We model the stochastic process for the default value as follows. The government almost always has the option to default and receive a payoff of $V^D(t) = \underline{V}$. With constant arrival probability λ, this default value temporarily increases to $V^D(t) = \overline{V} > \underline{V}$. The higher value represents an opportunity to default with lower consequences for punishment. If the government does not exercise this high default value option when it arrives, the default value returns to $\underline{V}$ until the next arrival of $\overline{V}$.

To define preferences, let $\boldsymbol{c} = \{c(t)\}_{t\geq 0}$ denote a *deterministic* consumption stream that characterizes the government's consumption until default.[8] We assume linear flow utility, $u(c) = c$. This allows an explicit characterization of the equilibrium objects while incorporating key economic forces that are robust to curvature in utility. Consumption at each point in time is restricted to lie in the interval $[\underline{C}, \overline{C}]$. Let $\boldsymbol{C}$ denote the space of consumption sequences with $c(t) \in [\underline{C}, \overline{C}]$ for all t.[9]

Given a consumption sequence $\boldsymbol{c}$, we define the government's expected value as follows. Let T denote the time at which the government defaults, if ever, at the low outside default value. The value to the government of a consumption sequence $\boldsymbol{c}$, $V(t, \boldsymbol{c})$, is recursively defined by:

$$\begin{aligned} V(t, \boldsymbol{c}) &= \sup_{T \geq t} \Bigg\{ \left[\int_t^T e^{-\rho(s-t)} c(s) \mathrm{d}s + e^{-\rho(T-t)} \underline{V} \right] e^{-\lambda(T-t)} + \\ &\qquad \int_t^T \left[\int_t^s e^{-\rho(\tau-t)} c(\tau) \mathrm{d}\tau + e^{-\rho(s-t)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \right] \lambda e^{-\lambda(s-t)} \mathrm{d}s \Bigg\} \\ &= \sup_{T \geq t} \Bigg\{ \int_t^T e^{-(\rho+\lambda)(s-t)} c(s) \mathrm{d}s + e^{-(\rho+\lambda)(T-t)} \underline{V} + \\ &\qquad \lambda \int_t^T e^{-(\rho+\lambda)(s-t)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \mathrm{d}s \Bigg\}. \end{aligned} \tag{1}$$

The first line is the value absent the arrival of the high default outside option, where the probability that T is reached before the first arrival of the high outside option is $e^{-\lambda(T-t)}$. The inner integral in the second term is the value conditional on the high outside option first arriving at time $s < T$, which is then integrated over all possible $s \in [t, T)$. The second equality follows from straightforward integration. Standard methods verify that there is a unique bounded fixed point V that satisfies (1) given $\boldsymbol{c}$. From (1), we have immediately that $V(t, \boldsymbol{c}) \geq \underline{V}$ for all t and $\boldsymbol{c}$, as $T = t$ is always an option.

We make the following assumptions on the primitives of the environment:

Assumption 1. *(i) $\rho \geq r$; (ii) $y \geq \rho\overline{V}$; (iii) $\overline{C} > y$; (iv) $\underline{C} < (\rho + \lambda)\underline{V} - \lambda\overline{V}$.*

The first item ensures that the government is relatively impatient (as compared to the market interest rate) and does not accumulate infinite assets. The second states that consuming the endowment forever is weakly greater than the high default value. If $\overline{V}$ were strictly greater than this value, the government may prefer default to holding a small amount of assets. When $y > \rho\overline{V}$, there is a deadweight loss in default; in particular, from the lenders' perspective, all debt is zeroed

[8] The fact that $c(t)$ is not indexed to the realization of V^D anticipates the fact that without contingent bonds, consumption will be deterministic conditional on no default.

[9] Given our restriction that $c \in [\underline{C}, \overline{C}]$, it would be equivalent to define u for the entire real line but set $u(c) = \overline{C}$ for $c \geq \overline{C}$ and $u(c) = -\infty$ for $c \leq \underline{C}$.

out once default occurs, but the government receives a value that is strictly less than full debt forgiveness. In the original Eaton and Gersovitz (1981), this difference reflected the loss of insurance. In the recent quantitative literature starting with Aguiar and Gopinath (2006) and Arellano (2008), an additional endowment cost is imposed during default. In the current environment, the gap $y - \rho\overline{V}$ makes default inefficient (in terms of joint borrower-lender surplus) and will play an important role in equilibrium debt dynamics. The third condition ensures that consuming the endowment is always feasible. The final condition guarantees that it is feasible to deliver the low default value to the government, $\underline{V}$, by assigning it a sufficiently low consumption level and letting the government default once $V^D(t) = \overline{V}$. That is, $\underline{V}$ is feasible without requiring an immediate default.

Some of the assumptions above were made to obtain tractability and build on earlier work in this literature.[10] However, as we will see in Section 7 the underlying economics are robust to the inclusion of endowment risk, concave utility, and discrete time.

3 Constrained Efficient Allocations

We first study an efficient allocation that maximizes the joint surplus between a risk-neutral lender and the government subject to the government's lack of commitment to repay. The efficient allocations provide a useful benchmark to understand the competitive equilibria studied in the next section.

Consider a Pareto planning problem that maximizes the expected payments to a risk-neutral lender conditional on delivering a value weakly greater than v to the government. As in Aguiar et al. (2018), the planning problem chooses a consumption stream $\boldsymbol{c}$, but the planner cannot prevent the government from defaulting when the government finds it optimal to do so. In particular, for consumption sequence $\boldsymbol{c}$, the government's value is defined by (1). When the government is indifferent to default or continuing, the planner can break the tie.

Given an allocation $\boldsymbol{c}$ and time T that maximizes (1) at time $t = 0$, the expected payments to the lender can be defined as:

$$P(\boldsymbol{c}, T) = \int_0^T e^{-\int_0^t r + \mathbb{1}_{[V(s,\boldsymbol{c})<\overline{V}]}\lambda ds}[y - c(t)]\mathrm{d}t, \tag{2}$$

where $\mathbb{1}_{[x]}$ is an indicator function that takes value one if x is true and zero otherwise. The inte-

[10] For continuous time formulations of sovereign debt models with Poisson shocks that trigger default see Aguiar, Amador, Farhi and Gopinath (2013), Lorenzoni and Werning (2013) and Bornstein (2018). For Brownian motion shocks, see Nuño and Thomas (2015), Tourre (2017), and DeMarzo, He and Tourre (2018). The latter two also discuss how linear utility (in their cases without consumption bounds) facilitates finding closed form solutions. See also Carré, Cohen and Villemot (2019) for Lévy processes.

grand represents the flow payments to the lender, which are discounted by r and the probability of default prior to period T. Here we have incorporated that the government does not default when indifferent upon the arrival of the high default value, which is without loss given that we will focus on Pareto efficient allocations.

Definition 1. *An allocation* $\{\boldsymbol{c}, T\}$ *is* efficient *if* T *maximizes* (1) *at* $t = 0$ *given* $\boldsymbol{c}$*, and if there is no alternative allocation* $(\tilde{\boldsymbol{c}}, \tilde{T})$ *such that* $V(0, \tilde{\boldsymbol{c}}) \geq V(0, \boldsymbol{c})$ *and* $P(\tilde{\boldsymbol{c}}, \tilde{T}) \geq P(\boldsymbol{c}, T)$*, with one inequality strict.*

Toward characterizing efficient allocations, we define the following planning problem:

$$P^\star(v) = \sup_{\boldsymbol{c} \in \boldsymbol{C}, T \geq 0} P(\boldsymbol{c}, T) \tag{3}$$
$$\text{subject to } \begin{cases} V(0, \boldsymbol{c}) = v \\ T \text{ maximizes (1) at } t = 0. \end{cases}$$

We define $P^\star$ on the domain $v \in [\underline{V}, V_{max}] \equiv \mathbb{V}$. It is infeasible to deliver $v < \underline{V}$. It is also infeasible to deliver higher value than $\overline{C}/\rho$, and we assume $V_{max} < \overline{C}/\rho$.[11] Note that if $P^\star$ is strictly decreasing, it characterizes the Pareto frontier. In what follows, we assume $\underline{C}$ is sufficiently low to guarantee that $P^\star$ is strictly decreasing.[12]

The first result states that we can restrict attention to allocations in which default occurs only if $V^D(t) = \overline{V}$:

Lemma 1. *It is weakly optimal in problem* (3) *to never default if* $V^D(t) = \underline{V}$*. That is, in any efficient allocation,* $T = \infty$*.*

The argument why default never occurs at $V^D(t) = \underline{V}$ is as follows. It is always feasible to deliver $\underline{V}$ by choosing a constant level of consumption until the first arrival of $\overline{V}$, at which point the government defaults (a result that follows from Assumption 1(iv)). This level of consumption is strictly less than the endowment because of the deadweight costs of default at $\underline{V}$ (from Assumption 1(ii) and $\underline{V} < \overline{V}$). Hence, this allocation dominates immediate default at $\underline{V}$. This lemma allows us to substitute $T = \infty$ in (3).

[11] The fact that V_{max} is strictly less than $\overline{C}/\rho$ ensures that the planner can set $\dot{v} < 0$ at the upper bound of the domain, a controllability requirement used in some of our proofs.

[12] The reason why $P^\star(v)$ may not be decreasing is that the threat of default is so severe that the planner would rather "forgive debt" by raising v to $v' > v(0)$ instantaneously at $t = 0$ without compensating lenders. If $\underline{C}$ is sufficiently low, forgiveness is dominated by setting $c = \underline{C}$ until $v(t) = v'$. As $\underline{C} \to -\infty$, this approximates a lump-sum payment at $t = 0$, which allows the planner to move v arbitrarily fast relative to the first arrival of $\overline{V}$. Specifically, $\lim_{\underline{C} \to -\infty}(P^\star(v) - P^\star(v')) \geq v' - v$.

To solve problem (3), we appeal to standard recursive techniques and study the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(r + \mathbb{1}_{[v<\overline{V}]}\lambda)P^{\star}(v) = \sup_{c\in[\underline{C},\overline{C}]} \left\{y - c + P^{\star\prime}(v)\dot{v}\right\}, \tag{P}$$

subject to

$$\dot{v} = -c + \rho v - \mathbb{1}_{[v<\overline{V}]}\lambda \left[\overline{V} - v\right] \tag{4}$$

and the state-space constraint $v \in \mathbb{V}$. Let $C^{\star}(v)$ denote an optimal policy associated with this recursive formulation. Proposition B.1 in the appendix details the necessary and sufficient conditions for a candidate value function to be a solution to (P).

Problem (P) implies that we can divide the state space into two regions. For $v \in [\underline{V}, \overline{V})$, default occurs with probability λ. Following Cole and Kehoe (2000), we refer to this subset of the domain as the *Crisis Zone*. For $v \in [\overline{V}, V_{max}]$, default does not occur even if the high outside default value is available. We refer to this subset as the *Safe Zone*. The fact that default occurs in the Crisis Zone even in the presence of a deadweight loss (that is, $\rho\overline{V} < y$) reflects market incompleteness. Specifically, the planner would like to adjust consumption in response to the realization of $\overline{V}$, but is prevented from doing so.

To characterize Pareto efficient allocations, we proceed in steps: we conjecture a candidate efficient allocation; we solve (P) under this conjecture; and then we verify if and when the candidate allocation satisfies the optimality conditions set out in Proposition B.1. Our conjectures are guided by the two competing forces driving debt dynamics; namely, relative impatience favors debt accumulation, while the costs of default favor debt reduction. The next two subsections derive solutions assuming that the borrowing and saving forces dominate, respectively. With the solutions in hand, we verify under what parameter configurations they solve the Pareto problem. We also verify that there are no parameter configurations for which neither the borrowing nor the saving allocation is efficient.

3.1 Efficient Borrowing Allocations

We first conjecture that the borrowing incentive dominates. Given the linearity of utility, a reasonable conjecture is that consumption is at the upper bound until v reaches $\underline{V}$. In particular, we define

$$C^{\star}_B(v) \equiv \begin{cases} \overline{C} \text{ for } v \in (\underline{V}, V_{max}] \\ (\rho+\lambda)\underline{V} - \lambda\overline{V} \text{ for } v = \underline{V}. \end{cases} \tag{5}$$

This sets consumption at its maximum possible level, $\overline{C}$, for the entire state space except at the lowest possible value $\underline{V}$. This implies $\dot{v} < 0$ for $v > \underline{V}$. At $v = \underline{V}$, the value cannot be further reduced given the government's option to default. Hence, consumption is set to deliver $\dot{v} = 0$. From equation (4), $\dot{v} = 0$ at $\underline{V}$ implies that $c = (\rho+\lambda)\underline{V} - \lambda\overline{V}$.

Let $P^{\star}_B$ denote the value to the lenders under this conjectured consumption policy function. We solve for $P^{\star}_B$ using (P) together with the value at the boundary, $P^{\star}_B(\underline{V})$ (which is determined by consumption at $\underline{V}$). The appendix contains closed form expressions for $P^{\star}_B$ for finite $\overline{C}$; in the text, we take the limit as $\overline{C} \to \infty$ to provide intuition. In particular, for any v,

$$\lim_{\overline{C}\to\infty} P^{\star}_B(v) = P^{\star}_B(\underline{V}) - (v - \underline{V}), \tag{6}$$

where $P^{\star}_B(\underline{V}) = (y - (\rho+\lambda)\underline{V} + \lambda\overline{V})/(r+\lambda)$. Expression (6) states that the payment to the lenders is the maximal incentive-compatible payment minus a lump sum consumed by the government in the initial period.

We now verify if and when $P^{\star}_B$ is a solution to Problem (P). Consider an initial promised value at the boundary of the Safe Zone, $v = \overline{V}$. One feasible allocation is to set consumption at $c = \rho\overline{V}$. This maintains a constant value of $\overline{V}$ for the government, which guarantees no default. The value to the lender is $(y - \rho\overline{V})/r$. A necessary condition for $P^{\star}_B$ to be optimal is that it delivers weakly greater value at $\overline{V}$ than this alternative. We show that this is also sufficient:

Proposition 1. *$P^{\star}_B$ is a solution to the planning problem if and only if*

$$rP^{\star}_B(\overline{V}) \geq y - \rho\overline{V}. \tag{7}$$

This condition has the following interpretation: it is efficient to borrow into the Crisis Zone rather than remain in the Safe Zone indefinitely. The left-hand side is the annuitized value of the objective from borrowing into the Crisis Zone. The right-hand side is the net payments to the lender from setting $\dot{v} = 0$ at the boundary of the Safe Zone (that is, the payments that guarantee that the government's value does not enter the Crisis Zone). The decision of whether to exit the Safe Zone is the crucial question given the inefficiencies associated with default, and the proposition states that this is the only restriction on parameter values that needs to be checked to verify that the borrowing allocation is efficient.

Again, for intuition, we let $\overline{C} \to \infty$, and (7) becomes:

$$r(\rho - r)\left(\overline{V} - \underline{V}\right) \geq \lambda\left(y - \rho\overline{V}\right). \tag{8}$$

The right-hand side represents the deadweight costs of default times the probability of default in the Crisis Zone. The larger this is, the more costly it is to enter the Crisis Zone and the more stringent this condition. The left-hand side captures relative impatience and the value of delivering utility to the government by front-loading consumption. The larger the discount rate ρ, the less stringent this condition. When $\rho = r$, the condition cannot be satisfied if $y > \rho\overline{V}$, that is, if there is a deadweight cost to default and the government is not impatient. Conversely, if $y = \rho\overline{V}$, this condition will be satisfied as long as $\rho > r$. This logic extends to finite $\overline{C}$.

Note that if condition (8) is violated (or more generally, condition (7)), then the planner would not find it optimal to borrow into the Crisis Zone: it prefers to deliver $\overline{V}$ to the government without inducing a future default. We use this observation to construct our second type of efficient allocations, where the planner chooses to exit the Crisis Zone when the promised value is close to the boundary of the Safe Zone.

3.2 Efficient Saving Allocations

An alternative to borrowing into the Crisis Zone is to save into the Safe Zone. This allocation favors reducing the probability of default over the relative impatience of the government.

We start then by conjecturing that the Safe Zone is an absorbing state. In particular, for the Safe Zone, we let consumption be

$$C_S^{\star}(v) \equiv \begin{cases} \overline{C} & \text{if } v \in (\overline{V}, \overline{C}/\rho) \\ \rho\overline{V} & \text{if } v = \overline{V}. \end{cases} \tag{9}$$

This implies that in the interior of the Safe Zone, the government receives the maximal consumption. However, at the boundary, the government receives the consumption that sets $\dot{v} = 0$, and hence v never transits from the Safe Zone into the Crisis Zone. With this conjecture, we can solve for the implied value function in the Safe Zone, which we denote $P_S^{\star}$ (see the appendix for the closed form expression).

For the Crisis Zone, the planner decides between saving toward the Safe Zone or remaining in the Crisis Zone. We denote the former scenario with a "hat." In particular, the linearity of the problem leads us to conjecture that if saving is efficient, consumption will be at its lower bound.

Thus, we define

$$\hat{C}(v) \equiv \underline{C} \text{ for } v \in [\underline{V}, \overline{V}). \tag{10}$$

The associated value from this policy is $\hat{P}$, which is obtained by solving (P) using $P_S^\star(\overline{V})$ as a boundary condition.

The appendix contains the expression for $\hat{P}$ for finite $\underline{C}$; for intuition, we take the limit as saving becomes arbitrarily fast:

$$\lim_{\underline{C} \to -\infty} \hat{P}(v) = P_S^\star(\overline{V}) + \overline{V} - v. \tag{11}$$

That is, the conjectured allocation calls for an initial lump sum payment by the government that is sufficient to reach the boundary of the Safe Zone immediately.

The value from saving into the Safe Zone is one building block of the efficient saving allocation. However, the planner may find it optimal to abandon the savings strategy in the Crisis Zone and instead pursue the borrowing one when the initial debt level is sufficiently high (that is, the promised is low). As a result, our conjectured value function in the Crisis Zone is the upper envelope of the savings and the borrowing conjectures:[13]

$$P_S^\star(v) \equiv \max\langle \hat{P}(v), P_B^\star(v) \rangle \text{ for } v \in [\underline{V}, \overline{V}). \tag{12}$$

It is possible to show that $\hat{P}$ and $P_B^\star$ cross at most once for $v \in [\underline{V}, \overline{V}]$. We denote by $v^I \in [\underline{V}, \overline{V}]$ such a crossing point and set $v^I = \underline{V}$ if they do not cross. The point v^I has a particular interpretation: the planner is indifferent between saving out of the Crisis Zone versus remaining in the Crisis Zone indefinitely at that point. For values of v above v^I, the planner finds it optimal to save, while for values below v^I, the planner finds it optimal to borrow. With this result in hand, we can complete the characterization of the policy function by setting

$$C_S^\star(v) \equiv \begin{cases} \underline{C} & \text{if } v \in [v^I, \overline{V}) \\ C_B^\star(v) & \text{if } v \in [\underline{V}, v^I). \end{cases} \tag{13}$$

To verify that $P_S^\star$ is a solution to the planning problem, again consider the point $\overline{V}$ at the boundary of the Safe and Crisis Zones. As with the borrowing allocation, the crucial condition is whether at the boundary of the Safe Zone, the objective is maximized by staying put versus borrowing to the upper bound:

[13]The Cole-Kehoe model also features a savings and a borrowing region within the Crisis Zone (for certain parameter values) when the government is impatient. See, for example, Cole and Kehoe (1996), Figure 2.

Proposition 2. *$P_S^\star$ is a solution to the planning problem if and only if*

$$rP_S^\star(\overline{V}) = y - \rho\overline{V} \geq rP_B^\star(\overline{V}). \tag{14}$$

Note that this condition is the mirror image of Proposition 1, which established the efficiency of the borrowing allocation. Together, Proposition 1 and 2 characterize an efficient allocation under any parameter configuration consistent with Assumption 1. In particular, either the borrowing or the saving allocation is efficient.

4 Competitive Equilibria

We now discuss competitive equilibria, and, as we will see, the efficient allocations provide a useful benchmark in the characterization.

We consider Markov equilibria. The payoff relevant states are the face value of debt b and default payoff V^D. Recall that the high default payoff state is only relevant if the government exercises the option to default; otherwise, the low default payoff state resumes. Therefore, we subsume the notation for the default payoff state $V^D = \underline{V}$ when defining prices and values conditional on repayment.

4.1 The Government's Problem

Let $V(b)$ denote the government's equilibrium value of repayment given the face value of debt b. Strategic default implies repayment if $V(b) \geq V^D$, and default otherwise.

Parallel to the analysis of Section 3, it is useful to split the state space into two regions. Given an equilibrium value V, we define the following: the *Safe Zone* is $b \in [-\overline{a}, \underline{b}]$ where $\underline{b}$ satisfies $V(\underline{b}) = \overline{V}$ and define $\overline{a} \equiv (\overline{C} - y)/r$ as the upper bound on assets that can be consumed; and the *Crisis Zone* is $b \in (\underline{b}, \overline{b}]$, where $\overline{b}$ satisfies $V(\overline{b}) = \underline{V}$. In each of the equilibria we study, we will establish the existence of these thresholds. As in the preceding analysis, the Safe Zone is the space of debt (and assets) such that the government will not default if the high default payoff state arrives. However, the government may default at some point in the future. The Crisis Zone is the space of debt such that the government will default upon the arrival of $V^D = \overline{V}$. For $b > \overline{b}$, the debt level is so high that, if the initial state is in this region, the government defaults immediately regardless of the payoff state. This region is beyond the endogenous borrowing limit and will never be reached from below in equilibrium. We denote the relevant debt state space in a competitive equilibrium by $\boldsymbol{B} \equiv [-\overline{a}, \overline{b}]$.

To characterize the government's problem, assume that the government faces an equilibrium price schedule $q : \boldsymbol{B} \to [\underline{q}, 1]$, where $\underline{q} > 0$ is defined as

$$\underline{q} \equiv \frac{r+\delta}{r+\delta+\lambda}, \tag{15}$$

which is the lowest possible bond price consistent with equilibrium in $\boldsymbol{B}$ (that is, the bond price that obtains when the government always defaults at the first arrival of $\overline{V}$).

At each point in time, the government chooses consumption as well as decides whether to pay its debt obligations or default after observing the realized V^D. Given consumption c, the government's debt, conditional on repayment, evolves according to

$$q(b)[\dot{b} + \delta b] = c + (r+\delta)b - y, \tag{16}$$

where $\dot{b}$ denotes the derivative of debt with respect to time. The left-hand side represents revenue from bond auctions, where the term in brackets is the change in the face value of debt plus the fraction of debt that matured, which is net new issuances. The terms on the right represent consumption plus payments of interest and principal minus income.

It may be the case that $q(b)$ is discontinuous at some debt level b_0. This occurs, for example, when the government is indifferent between borrowing or saving. When indifferent, we break the tie by having the government save, which implies that the equilibrium price at b_0 is the highest of the prices consistent with the two possible strategies.[14]

We prove in Appendix Lemma B.2 that the government's value function, V, is strictly decreasing and Lipschitz continuous. In addition, it is the unique, bounded, continuous solution to the following HJB equation on $\boldsymbol{B}$, given a price schedule q:

$$(\rho + \Lambda(b))V(b) = \max_{c \in [\underline{C}, \overline{C}]} \left\{ c + V'(b) \underbrace{\left(\frac{c + (r+\delta)b - y}{q(b)} - \delta b \right)}_{\dot{b}} + \Lambda(b)\overline{V} \right\}, \tag{17}$$

[14]For technical reasons, we place one more constraint on debt issuance policies around points of price discontinuity. We impose that for an arbitrarily small neighborhood around b_0, debt buybacks occur at price approaching $q(b_0)$. The specifics are spelled out in Appendix C.5. Debt buybacks occur when $\dot{b} < -\delta b$, that is, when debt decreases faster than existing debt matures. Imposing that buybacks occur at the higher of the two prices around the discontinuity allows us to apply recent results in optimal control with discontinuous dynamics. Note that this condition is imposed only around points of discontinuity in the price schedule and for an *arbitrarily small* interval around them. We flag when we use this restriction in footnotes 28 and 31. In what follows, we suppress this constraint in the notation for the government's HJB equation.

where

$$\Lambda(b) \equiv \lambda \mathbb{1}_{[V(b)<\overline{V}]}. \tag{18}$$

In the appendix, Proposition B.2 provides necessary and sufficient conditions for a solution to (17).

4.2 The Lenders' Problem

The equilibrium condition from the lenders' problem is that lenders must be indifferent to purchasing the government's bonds versus holding risk-free assets that return R. We consider $b \leq 0$ to represent risk-free assets held abroad that have a price of one. For $b > 0$, b represents the liabilities of the government. To price debt in equilibrium, consider starting from a debt level $b > 0$, and using the government's policy $C(b)$ and the budget constraint (16) to derive the equilibrium path of debt going forward, $b(t)$. The present value "break-even" bond pricing equation for the lender is[15]

$$q(b) = \int_0^\infty e^{-(r+\delta)t - \int_0^t \Lambda(b(s))\mathrm{d}s}(r+\delta)\mathrm{d}t. \tag{19}$$

The integrand is the coupon payment r plus principal δ. The discount factor is the interest rate r plus the rate at which bonds mature δ plus a further discount to reflect the default survival probability. Note that given an equilibrium path $b(t)$, q satisfies the following ODE:

$$(r + \delta + \Lambda(b(t)))q(b(t)) = (r + \delta) + q'(b(t))b'(t). \tag{20}$$

for $b(t) > 0$.

4.3 Definition of Equilibrium

We are ready to define an equilibrium:

Definition 2. *An* equilibrium *consists of a compact domain $\boldsymbol{B}$ and functions of debt, $\{q, V, C\}$, such that: (i) given the government's consumption policy C and strategic default, lenders break even in expectation at prices q; (ii) given a price schedule q, the government's maximal value conditional on repayment is $V(b)$, which is achieved by consuming $C(b) \in [\underline{C}, \overline{C}]$; and (iii) for $b \in \boldsymbol{B} = [-\overline{a}, \overline{b}]$, $V(b) \geq \underline{V}$, with $V(\overline{b}) = \underline{V}$.*

[15] This equation anticipates that the government does not cross $b = 0$ more than once in a Markov equilibrium; and does not default with assets.

In the definition of equilibria, we require that $V(\overline{b}) = \underline{V}$. That is, $\overline{b}$ represents the maximal endogenous borrowing limit. We do this to eliminate the possibility of generating equilibria that depend on ad hoc borrowing limits.

Note that b is the face value of debt, which defines the government's promised payments absent default. The expected present value of payments to lenders in equilibrium is the market value of debt: $q(b)b$. This distinction is useful to bear in mind when comparing competitive equilibria to the Pareto problem studied in Section 3.

Mirroring the analysis of efficient allocations, we focus on two types of equilibria. In a *borrowing equilibrium*, the government borrows up to its borrowing limit $\overline{b}$ regardless of initial conditions. In particular, if the government starts in the Safe Zone (or with assets), it borrows into the Crisis Zone and eventually defaults. In a *saving equilibrium*, the Safe Zone is an absorbing state.[16]

4.4 The Borrowing Equilibrium

We denote equilibrium objects in the borrowing equilibrium with the subscript B; that is, C_B, V_B, q_B are the consumption, value, and price functions, respectively. Similarly, let $\underline{b}_B$ denote the threshold between the Safe and Crisis Zones, and $\overline{b}_B$ the endogenous upper bound on debt.

In the borrowing equilibrium, we conjecture that the government borrows to its endogenous debt limit. Given the linearity of preferences and weak impatience, a reasonable conjecture is that the government consumes at its upper bound until $b = \overline{b}_B$. At the debt limit, the government pays coupons and rolls over maturing bonds until the first arrival of $\overline{V}$, at which point it defaults.

This allocation is the same allocation as in the efficient borrowing allocation, a symmetry we use to streamline the derivation. In particular, the conjectured equilibrium delivers the same payoffs to lenders and government as in the efficient borrowing allocation. Given a price schedule q_B, the borrowing equilibrium payoffs to the lender are $q_B(b)b$ and the payoff to the government is $V_B(b)$. Hence, it is the case that:

$$P_B^\star\left(V_B(b)\right) = q_B(b)b, \tag{21}$$

for any $b \leq \overline{b}_B$.

To solve for $\overline{b}_B$, note that in the Crisis Zone, there is a constant hazard λ of default, and thus the price of the bond equals $\underline{q}$ defined in (15). At $b = \overline{b}_B$, the government is indifferent to default at $\underline{V}$; that is, $V_B(\overline{b}_B) = \underline{V}$. From (21), we have $\overline{b}_B \equiv P_B^\star(\underline{V})/\underline{q}$. Accordingly, we have $C_B(b) = \overline{C}$ for

[16]In Appendix A, we discuss a third type of Markov equilibrium, which we denote a *hybrid equilibrium* because it combines features of both the saving and borrowing equilibria. Given the multiplicity we discuss below, one could also construct sunspot equilibria.

$b < \overline{b}_B$, and $C_B(\overline{b}_B) = C^\star_B(\underline{V})$.

We solve for the boundary of the Safe Zone, $\underline{b}_B$, in a similar fashion. At the boundary, we have $P^\star_B(\overline{V}) = q_B(\underline{b}_B)\underline{b}_B = \underline{q}\underline{b}_B$, where the last inequality uses the knowledge of the price at the boundary under the conjectured borrowing dynamics.[17].

Given the price schedule in the Crisis Zone $(\underline{b}_B, \overline{b}_B]$, we extend q_B into the Safe Zone by solving the ODE in (20) with boundary condition $q_B(\underline{b}_B) = \underline{q}$. The solution can be expressed in closed form (see appendix equation (31)). For $b \in [-\overline{a}, 0]$, the equilibrium price is 1 (as the government has assets). Letting $\boldsymbol{B}_B \equiv [-\overline{a}, \overline{b}_B]$, this completes our conjecture of the borrowing equilibrium.

Figure 1 depicts the equilibrium objects for a parameterized borrowing equilibrium. Panel (a) depicts the value function. The dotted horizontal lines represent the two default values, $\underline{V}$ and $\overline{V}$. The Safe Zone is demarcated by the vertical line at $\underline{b}_B$. By definition, $V_B(\underline{b}_B) = \overline{V}$ at this point. Similarly, the endogenous upper bound of debt, $\overline{b}_B$, occurs when $V_B(b)$ intersects $\underline{V}$. For reference, the dashed line depicts the value of setting $\dot{b} = 0$, given the equilibrium price schedule and the equilibrium default policy. The stationary value has a discontinuity at $\underline{b}_B$ because defaulting when $\overline{V}$ arrives is strictly better than the stationary value. The stationary value is the same as the equilibrium value at the upper bound $\overline{b}_B$. Panel (b) of Figure 1 depicts the price schedule. The price is monotonically decreasing in the Safe Zone and then is flat at $\underline{q}$ for $b \in [\underline{b}_B, \overline{b}_B]$. The consumption policy function is depicted in Panel (c). For reference, the dashed line depicts the stationary consumption level, given the equilibrium price schedule. Consumption is strictly above the dashed benchmark until $b = \overline{b}_B$, at which point consumption drops to the stationary level.

To verify when the conjectured borrowing equilibrium satisfies the equilibrium conditions, we need to check that V_B is a solution of (17). In this case, the important condition is that starting from the Safe Zone, the government prefers to borrow into the Crisis Zone and eventually default rather than remain in the Safe Zone. We have:

Proposition 3. *The conjectured borrowing equilibrium $\{C_B, V_B, q_B, \boldsymbol{B}_B\}$ is a competitive equilibrium if and only if*

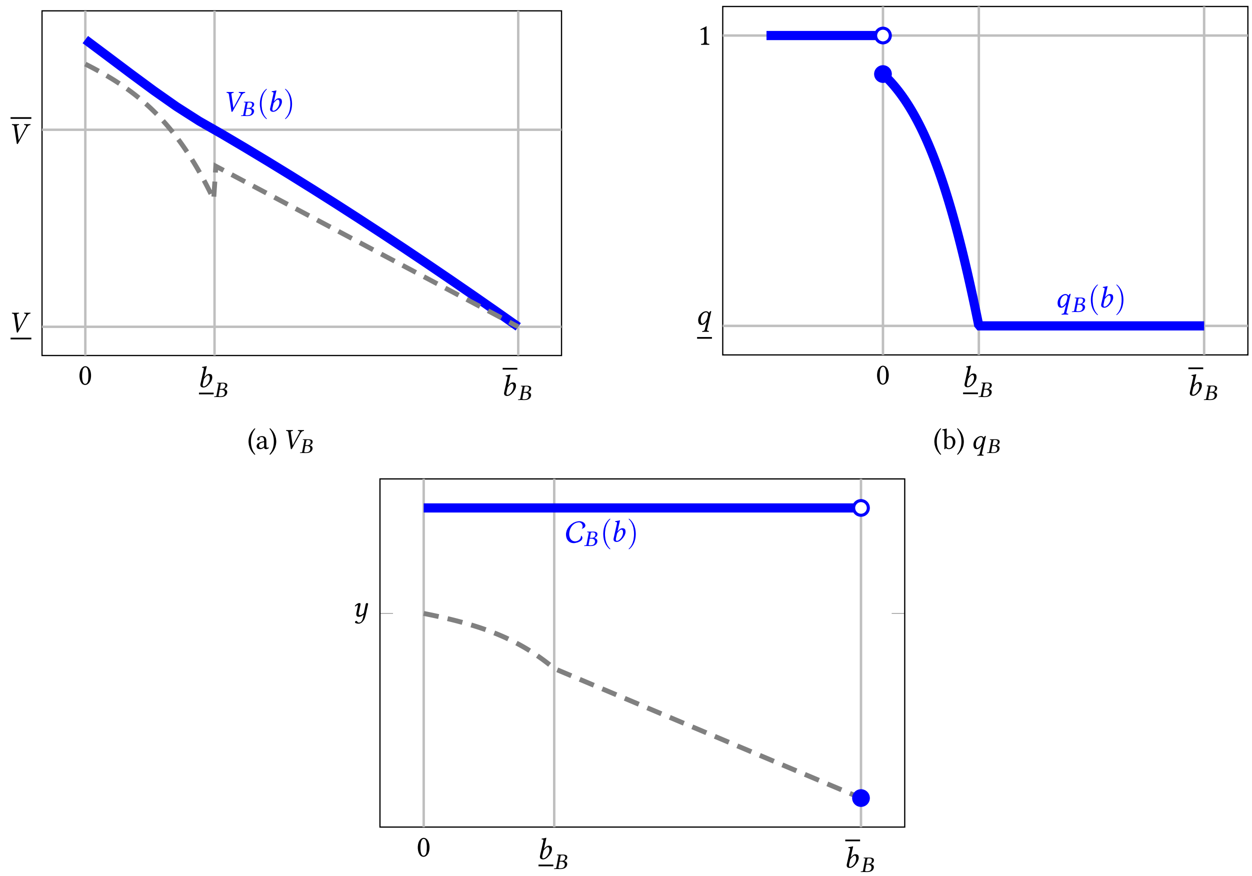
$$V_B(b) \geq \frac{y - [r + \delta(1 - q_B(b))]\, b}{\rho}, \text{ for all } b \in [0, \underline{b}_B]. \tag{22}$$

The right-hand side of (22) is the value of indefinitely consuming the stationary level of consumption at *equilibrium prices* in the Safe Zone. Thus, borrowing into the Crisis Zone is an equilibrium outcome if doing so dominates remaining in the Safe Zone.

Crucially, condition (22) is a weaker condition than for borrowing to be efficient, condition (7). Efficiency requires $P^\star_B(v) \geq (y - \rho v)/r$ in the Safe Zone. Using that the equilibrium payoff to

[17] Using the formula for $P^\star_B$ in the appendix, we can show that $\underline{b}_B \geq 0$ as $P^\star_B(\overline{V}) \geq 0$

Figure 1: Borrowing Equilibrium



(a) V_B

(b) q_B

(c) C_B

The figure depicts the value, price, and consumption functions in a borrowing equilibrium, respectively. The equilibrium functions are represented by the bold solid blue lines. The horizontal lines in the value function plots represent the two default values. The dashed line in the value function plots represents the stationary value function at the corresponding equilibrium prices. The dashed line in the consumption plots represents the level of consumption associated with the stationary value. The equilibrium is constructed with parameters $r = 1$, $\rho = 2$, $y = 1$, $\lambda = 2$, $\delta = 10$, $\overline{C} = 1.2$, $\underline{V} = .8y/\rho$, and $\overline{V} = .95y/\rho$.

lenders is $q_B(b)b$ and the government's value is $V_B(b)$, the efficiency condition (7) can be rewritten as:

$$V_B(b) \geq \frac{y - rq_B(b)b}{\rho}, \tag{23}$$

for all $b \in [0, \underline{b}_B]$. As $q_B(b) < 1$ on this domain, condition (22) is strictly weaker than (23).

Both inequalities (22) and (23) compare the value function to the value that would be generated by keeping the level of debt constant. The difference between the two inequalities is the price used to compute this stationary value. In inequality (22), the comparison uses the equilibrium prices. In equation (23), the comparison uses the planner's cost of rolling over the lender's value, $q_B(b)b$, at the risk-free interest rate r. This difference stems from a time consistency problem. The planner can commit to remaining in the Safe Zone, and in that case, discounts payments at the risk free rate r. In a borrowing equilibrium, the cost of keeping debt constant in the safe zone is strictly greater than r. In this equilibrium, lenders expect that the government in the future will borrow into the Crisis Zone and eventually default. If the government were to remain in the Safe Zone today, *because of these expectations with regard to its future behavior*, the price of the bonds would remain lower than one. Hence, it would nevertheless pay a default premium, rolling debt over at a yield greater than r. Thus, the crucial time consistency problem in the borrowing equilibrium is the inability to credibly commit not to exit the Safe Zone at some point in the future. The link between creditor beliefs about future fiscal policy and the government's best response to the resulting equilibrium price schedule will provide the source of multiplicity discussed in the next section.

Maturity is at the heart of this time consistency problem. To see this, let us consider what happens when $\delta \to \infty$, that is, as the bonds mature instantaneously (the appropriate continuous time analog of one-period debt). In the proof of the next proposition, we show that $q_B(b) \to 1$ and $\delta(1 - q_B(b)) \to 0$ for $b \in [0, \underline{b}_B)$, as $\delta \to \infty$. Hence, the equilibrium condition (22) and the efficiency condition (23) become identical.[18] More generally, the proof of the following proposition establishes that condition (22) becomes stronger as δ increases. Summarizing the above,

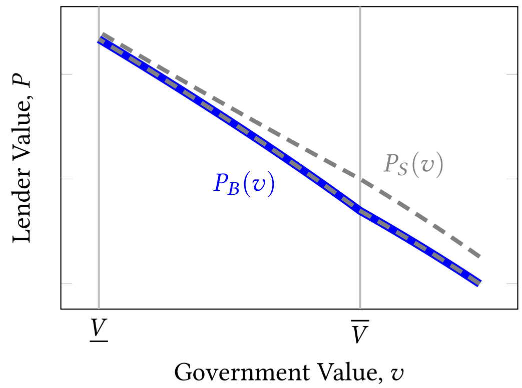
Proposition 4. *The following holds:*

(i) If the borrowing allocation is efficient, then the conjectured borrowing equilibrium is a competitive equilibrium for any δ;

(ii) If the borrowing equilibrium exists for δ_0, then it exists for any $\delta \in [0, \delta_0]$; and

[18]Note that δ has no effect on $P_B^\star$ because the planning problem is independent of maturity. Even though $\underline{b}_B$ is affected by changes in δ, $q_B(\underline{b}_B)\underline{b}_B$ remains constant.

Figure 2: Joint Surplus: Borrowing Equilibrium



The figure depicts the joint surplus in the borrowing equilibrium. The solid line is a parametric plot of $(V_B(b), q_B(b)b)$ for $b \in [0, \overline{b}_B]$. The dashed reference line is $P^\star(v)$ for $v \in [\underline{V}, V_B(0)]$. The parameters are the same as in Figure 1.

(iii) *If the borrowing allocation is* not *efficient, then there exists a* $\delta_1 < \infty$ *such that the conjectured borrowing equilibrium is* not *a competitive equilibrium for* $\delta > \delta_1$.

In Figure 2, we plot the market value of debt, $q_B(b)b$, against the corresponding value for the government, $V_B(b)$, using the same parameters as in Figure 1. Specifically, the solid line in the figure depicts the joint surplus between the lenders and the government in a competitive equilibrium. The dashed line is the efficient frontier, which in this parameterization is the efficient saving value, $P^\star_S(v)$. The efficient borrowing value, $P^\star_B$, is identical to the equilibrium frontier. The inefficiency of the borrowing equilibrium reflects that the government borrows in the competitive equilibrium, while the planner would like to implement the saving allocation.

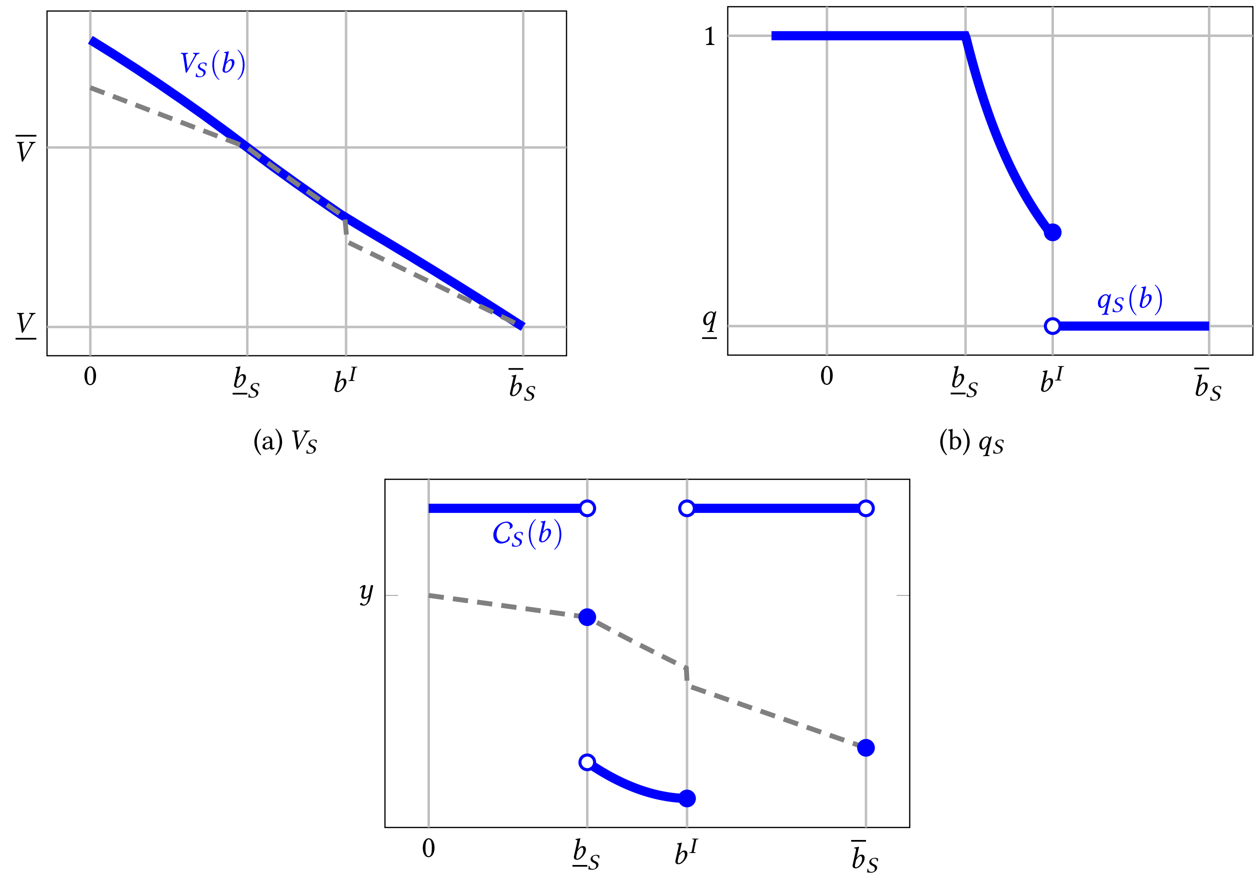
4.5 The Saving Equilibrium

We now consider an alternative equilibrium that features saving out of the Crisis Zone. The approach closely parallels that of the efficient saving allocation. As in the efficient saving allocation, we conjecture that the Safe Zone is an absorbing state and the Crisis Zone can potentially be divided into a saving region and a borrowing region. Let $[-\overline{a}, \underline{b}_S]$ denote the Safe Zone, $(\underline{b}_S, b^I]$ the saving region in the Crisis Zone, and $(b^I, \overline{b}_S]$ the borrowing region in the Crisis Zone.

As the Safe Zone is absorbing, prices are one for $b \leq \underline{b}_S$. The government's value at the boundary is $\overline{V}$ by definition, which is obtained by consuming $y - r\underline{b}_S$ forever. Thus, $\underline{b}_S = (y - \rho\overline{V})/r$.

For $b \in (\underline{b}_S, b^I]$, the government is actively saving toward the Safe Zone. The value and associated prices solve the two ODE's characterizing the government's and lenders' problems using the Safe Zone value and price as boundary conditions. We present the details and solutions in the appendix.

Figure 3: Saving Equilibrium



(a) V_S

(b) q_S

(c) C_S

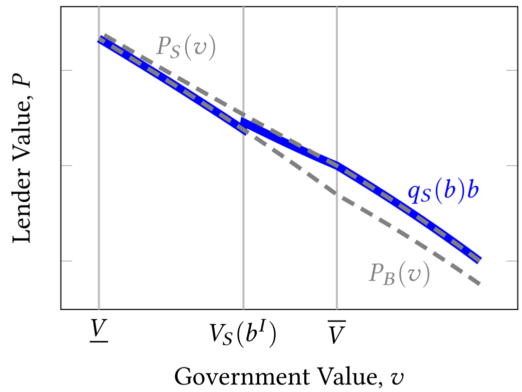
The figure depicts the value, price and consumption functions in a saving equilibrium, respectively. The equilibrium functions are represented by the bold solid blue lines. The horizontal lines in Panel (a) represent the two default values. The dashed line in Panel (a) represents the stationary value function at the corresponding equilibrium prices. The dashed line in Panel (b) represents the level of consumption associated with the stationary value. The equilibrium is constructed with the same parameters as Figure 1: $r = 1$, $\rho = 2$, $y = 1$, $\lambda = 2$, $\delta = 10$, $\overline{C} = 1.2$, $\underline{V} = .8y/\rho$, and $\overline{V} = .95y/\rho$. The value of $\underline{C}$ is set low enough so that it never binds in equilibrium.

The debt level b^I, if it exists, is determined as in the efficient analysis; specifically, it is the unique point of indifference between saving and borrowing in the Crisis Zone. For $b \in (b^I, \overline{b}_B]$, the value and prices are the same as in the borrowing equilibrium. If b^I does not exist, $\overline{b}_S$ is pinned down by the point the value of saving reaches $\underline{V}$.

The equilibrium objects $\{C_S, V_S, q_S, \boldsymbol{B}_S\}$ are detailed in the appendix and depicted in Figure 3, which follows the layout of Figure 1.

As in our discussion of efficient allocations, the key question is whether it is optimal to remain in the Safe Zone or borrow to the upper bound. Crucially, for the equilibrium, the question is now whether the government finds it *privately* optimal. The condition for saving to be a valid equilibrium outcome is stated in the following proposition:

Figure 4: Joint Surplus: Saving Equilibrium



The figure depicts the joint surplus in the saving equilibrium. The solid line is a parametric plot of $(V_S(b), q_S(b)b)$ for $b \in [0, \overline{b}_B]$. The upper and lower dashed reference lines are $P_S^\star(v)$ and $P_B^\star(v)$, respectively, for $v \in [\underline{V}, V_B(0)]$. The parameters are the same as in Figure 1.

Proposition 5. *The conjectured saving equilibrium $\{C_S, V_S, q_S, \boldsymbol{B}_S\}$ is a competitive equilibrium if and only if*

$$\underline{b}_S \equiv \frac{y - \rho\overline{V}}{r} \geq \underline{b}_B. \tag{24}$$

To see why saving can be an equilibrium outcome, first note that the government always has the option to remain in the Crisis Zone and wait for the high default option. As $\overline{V} > V_S(b)$ in the Crisis Zone, this is a plausible alternative. The cost of this strategy is that the government must roll over its debt at a discounted price while waiting for $\overline{V}$. If instead the government saves to the Safe Zone, it can roll over its debt at the risk-free price. This increase in price ensures that the government at least partially internalizes the gain from reducing the probability of default and provides the government with the incentive to save.

However, the government's private incentive to save in equilibrium is weaker than that of the planner. Recall from Proposition 2 that saving is efficient if $(y - \rho\overline{V})/r \geq P_B^\star(\overline{V}) = \underline{q}\underline{b}_B$. As $\underline{q} < 1$, condition (24) is *stronger* than the efficiency condition. Thus, *efficiency of saving does not imply that it can be sustained in equilibrium.* That is, a necessary but not sufficient condition for a saving equilibrium to exist is that the saving allocation is efficient.

To gain more intuition, we let $\overline{C} \longrightarrow \infty$ and condition (24) becomes

$$r(\rho - r)\left(\overline{V} - \underline{V}\right) \leq \left[\frac{\delta}{r + \delta + \lambda}\right]\lambda(y - \rho\overline{V}).$$

The term in square brackets is strictly less than one. Comparing to condition (8), we see that the condition for saving in equilibrium is strictly tighter. We can also infer the role of maturity, δ.

The gap between the two conditions is decreasing in δ. As $\delta \to \infty$, that is when bonds have arbitrarily short maturity, then savings is an equilibrium if and only if it is efficient. As maturity lengthens, saving is harder to sustain in equilibrium even when efficient. In particular, the greater the fraction of debt rolled over each period, the stronger the government's private incentive to save, while maturity is irrelevant for the efficient allocation. At one extreme, if $\delta = 0$ and bonds are perpetuities, the government never saves in equilibrium regardless of efficiency; at the other extreme, as $\delta \to \infty$, the conditions for saving to be efficient and to be an equilibrium outcome converge. Collecting results:

Proposition 6. *A necessary condition for $\{C_S, V_S, q_S, \boldsymbol{B}_S\}$ to be a competitive equilibrium is that saving is efficient. If saving is strictly efficient, that is, $P_S^\star(\overline{V}) > P_B^\star(\overline{V})$, there exists a $\delta_S \in [0, \infty)$, defined by*

$$\delta_S \equiv \frac{\lambda P_B^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} - r, \tag{25}$$

such that $\{C_S, V_S, q_S, \boldsymbol{B}_S\}$ is a competitive equilibrium if $\delta \geq \delta_S$, and is not an equilibrium otherwise. If $\rho > r$, then $\delta_S > 0$.

Even when the government saves in equilibrium, it exits the Crisis Zone at a slower pace than the planner. In the planning problem, consumption is at its lower bound $\underline{C}$ in the saving region. In the appendix, we solve for equilibrium consumption and show that it is interior in the saving region. In particular, equilibrium debt dynamics are such that $\dot{b} \geq -\delta b$. That is, while saving, the government never repurchases non-matured bonds; it deleverages by letting bonds mature and not fully replacing them with new bonds. This reflects the inefficiency of long-term debt discussed by Aguiar et al. (2018). The government does not capture the full return to eliminating the probability of default and thus does not have an incentive to save as quickly as possible. This leads to a divergence between the saving equilibrium allocation and the efficient saving allocation.

Figure 4's solid line plots the market value of debt, $q_S(b)b$, against the government's value, $V_S(b)$. The upper and lower dashed lines are the efficient frontier for the saving and borrowing allocation, respectively. The saving allocation dominates the borrowing allocation and hence represents the Pareto frontier. For $v \in [\underline{V}, V_S(b^I)]$, that is, for $b \in (b^I, \overline{b}_B]$, the government borrows when it is efficient to save. For $v \in [V_S(b^I), \overline{V}]$, or $b \in [\underline{b}_S, b^I]$, the government saves, but at a rate that is inefficiently slow. Hence the equilibrium surplus remains within the Pareto frontier. Note that the discontinuity in the equilibrium price schedule at b^I is reflected in the sharp change in the lender's value around this threshold. For $v \geq \overline{V}$, or $b \leq \underline{b}_S$, the government is in the Safe Zone, and the efficient and equilibrium allocations coincide.

The fact that maturity drives a wedge between efficiency and equilibria anticipates the next section. Even when saving is efficient and can be supported as an equilibrium, it is still possible that the borrowing allocation remains a valid competitive equilibrium. In the next section, we discuss the role of maturity in this multiplicity.

5 Maturity and Multiplicity

The preceding section provided necessary and sufficient conditions for both the borrowing and saving equilibria. This allows us to explore under what parameterizations the model has multiplicity as well as the economics behind the multiplicity.

The key condition to sustain either equilibrium is whether the government prefers to remain in the Safe Zone or borrow into the Crisis Zone. Importantly, the government makes this decision taking the equilibrium price schedule as given. This is the crucial distinction between the equilibrium problem and the planning problem and is at the heart of the potential multiplicity.

First, consider the borrowing equilibrium depicted in Figure 1. While in the Safe Zone ($b < \underline{b}_B$), there is no threat of immediate default as $V_B(b) \geq \overline{V}$. Nevertheless, the bond price lies strictly below one. The creditors require a default premium because they anticipate that the government will borrow into the Crisis Zone ($b > \underline{b}_B$), and then potentially default, before the debt matures. Hence, the government does not have the option to remain in the Safe Zone at risk-free prices. Rather, the question is whether to maintain its debt position in the Safe Zone at a price below one, or borrow into the Crisis Zone. As can be seen, the stationary value in the Safe Zone lies strictly below the equilibrium value function. Given that the price schedule offers no reward for remaining in the Safe Zone, the creditors' pessimistic expectations become self-fulfilling.

Now consider the saving equilibrium depicted in Figure 3, constructed with the same parameter values. Note that the equilibrium price is one throughout the Safe Zone and then declines in the Crisis Zone. This nonlinearity in the price schedule is reflected in the government's value function. The payoff to saving out of the Crisis Zone is the high price at the boundary of the Safe Zone.[19]

Interestingly, across the two equilibria, the government borrows when prices are low (spreads are high), while it saves when prices are high (spreads are low). The important element of the price schedule is not the level, but the incentives or disincentives to borrow. In the saving equilibrium, the price schedule declines steeply once the government enters the Crisis Zone. In the borrowing equilibrium, the price schedule is flat at the boundary of the Safe Zone. In this way, the self-fulfilling dynamics we uncover in this paper provide an alternative view of the "gambling for redemption" hypothesis that explains the debt accumulation of debt-distressed European coun-

[19]Note that because $q_S(b) \geq q_B(b)$ for all $b \in \boldsymbol{B}_S \cap \boldsymbol{B}_B$ and $\boldsymbol{B}_B \subseteq \boldsymbol{B}_S$, the government always prefers to face the saving equilibrium price schedule.

tries during the debt crises (see Conesa and Kehoe, 2017). In our model, low debt prices and debt accumulation both arise endogenously.

Note that the multiplicity in the model is *dynamic* in that it depends on expectations of future equilibrium behavior. In particular, the equilibria are supported by different expectations about whether the government will borrow or save, and whether bond prices will be the risk-free price or something lower. The underlying tension is between the incentive to dilute long-term bondholders versus the incentive to economize on rollover costs. Which effect dominates in equilibrium depends on beliefs in a non-trivial part of the parameter space. Moreover, these competing forces highlight why maturity plays a central role in the existence of multiple equilibria.

For the limiting case of arbitrarily large $\overline{C}$, we can state a simple condition that determines when it is possible for both equilibria to be supported:

Proposition 7. *If the parameters satisfy the following condition:*

$$1 + \rho \left(\frac{\overline{V} - \underline{V}}{y - \rho \overline{\overline{V}}} \right) > \frac{\lambda}{\rho - r} > r \left(\frac{\overline{V} - \underline{V}}{y - \rho \overline{\overline{V}}} \right), \tag{26}$$

there exists an M and a non-empty interval $\Delta \subset [0, \infty)$, such that for all $\overline{C} > M$ and all $\delta \in \Delta$, both the borrowing and saving equilibria exist.

The second inequality in (26) guarantees that the saving allocation is efficient for arbitrarily large $\overline{C}$. We know from Proposition 6 that this is a necessary condition and sufficient for high enough δ for the saving equilibrium to exist.

The first inequality in (26) guarantees the existence of the borrowing equilibrium, for any finite δ, when $\overline{C}$ becomes arbitrarily large. When $\overline{C}$ becomes arbitrarily large, the price of the bond converges to $\underline{q}$ throughout the Safe Zone, as the rate at which the government exits the Safe Zone becomes arbitrarily fast. The first inequality verifies that the government prefers to borrow into the Crisis Zone when facing a price close to $\underline{q}$ for all debt levels in the Safe Zone.

This proposition shows that multiplicity is an endemic feature of this model when the government is impatient and there are deadweight losses from default.[20] That is,

Corollary 1. *If $\rho > r$ and $y > \rho \overline{V}$, there always exists a triplet $\{\delta, \lambda, \overline{C}\}$ such that both savings and borrowing equilibria exist.*

[20] In a discrete time version of the environment, we computed the model by backward induction assuming a finite horizon, T, and then letting T become very large. Holding constant the underlying parameters, such a procedure can converge to either a saving or borrowing equilibrium, depending on the maturity chosen. Hence, taking the limit of a finite horizon economy cannot be used to consistently select a particular type of equilibria.

6 Third-Party Policies

The existence of multiple equilibria raises the question of whether a deep-pocketed third party, such as the IMF or ECB, could induce market participants to play the preferred equilibrium. In the rollover crisis model of Cole and Kehoe (2000), a price floor would eliminate the crisis equilibrium. Similarly, in a Calvo-style crisis, a price floor (or a cap on spreads) would also eliminate the bad equilibrium. More importantly, such a policy would require no resources along the equilibrium path, as long as they were credible off equilibrium.

A natural policy question in our framework is how to prevent coordination on the borrowing equilibrium when saving is efficient. Debt forgiveness does not select a particular equilibrium because both equilibria co-exist at low debt levels. Hence, in the borrowing equilibrium, debt forgiveness provides only a temporary reduction in debt levels, as in the debt-overhang model of Aguiar and Amador (2011). Similarly, a price floor does not eliminate the inefficient equilibrium. In particular, with a lower bound on prices greater than $\underline{q}$, the borrowing equilibrium remains an equilibrium and the government would borrow up to its borrowing limit at the better price. The policy not only would fail, but also would cost resources along the equilibrium path.

More formally, consider a parameterization such that both saving and borrowing equilibria exist, with subscripts B and S denoting the respective equilibrium objects, as before. This parameterization is the natural launching point for policy intervention.

The intervention we study involves a third party that is willing to purchase government bonds at a price $q^\star$ as long as $b \leq b^\star$. This combines a price floor with a quantity restriction. To highlight the role of the price floor versus the quantity restriction, we consider two polar cases. In our first scenario, let $b^\star = \overline{b}_B$. That is, the quantity restriction is not tighter than the endogenous borrowing limit in the borrowing equilibrium. The second scenario sets $b^\star = \underline{b}_S$. This is a tight quantity restriction, designed such that interventions potentially involve only risk-free debt.

Let the superscript P indicate equilibrium objects in the presence of the third-party policy. The break-even condition for foreigners is

$$q^P(b) = \sup_{T\geq 0}\left\{\int_0^T e^{-(r+\delta)t-\int_0^t \Lambda^P(b^P(s))\mathrm{d}s}(r+\delta)\mathrm{d}t + e^{-(r+\delta)T}\mathbb{1}_{[b^P(T)\leq b^\star]}q^\star\right\}, \tag{27}$$

where $b^P(s)$ denotes the equilibrium evolution of bonds, starting from b, under the third-party policy. The equation captures that an investor considers the best among all possible hold-and-sell strategies: after purchasing the bonds, the investor can hold them up to any time T, at which point, if the total debt remains below $b^\star$, the investor has the option to sell them to the third party for a price of $q^\star$. Note that the assumption that all the investors are identical means we do not need to consider the strategies where one investor sells to another.

Given the price schedule, the problem of the government continues to be characterized by the HJB (17). As a result, in any equilibrium, there will be a Safe Zone and a Crisis Zone, demarcated by $\{\underline{b}^P, \overline{b}^P\}$, with $V^P(\underline{b}^P) = \overline{V}$ and $V^P(\overline{b}^P) = \underline{V}$.

As in the analysis without the third party, we will consider two equilibrium conjectures: a borrowing one and a saving one. Similarly to our benchmark analysis, in a conjectured borrowing equilibrium, starting from a debt level in the Safe Zone, the debt eventually reaches the Crisis Zone. In a conjectured saving equilibrium, the Safe Zone is an absorbing state.

Consider first the case where $b^\star = \overline{b}_B$. In this case, the policy does not eliminate the borrowing equilibrium. But if it is generous enough (that is, if $q^\star$ is high enough), then it eliminates the saving equilibrium:

Proposition 8 (Loose quantity restriction)**.** *Assume the inequalities in Proposition 7 are satisfied and $\overline{b}_B > \underline{b}_S$. Suppose $q^\star \in (\underline{q}, 1]$ and $b^\star = \overline{b}_B$, and let $\overline{C}$ be sufficiently large. Then,*

(i) There always exists a borrowing equilibrium. That is, there is an equilibrium where $C^P(b) = \overline{C}$ for all $b < b^\star$. In this equilibrium, the third party incurs losses.

(ii) There is a $\tilde{q} < 1$ such that for all $q^\star > \tilde{q}$, the saving equilibrium does not exist.

A better policy is to impose a tighter quantity restriction, that is, $b^\star = \underline{b}_S$. In this case, the policy selects the saving equilibrium for high enough $q^\star$:

Proposition 9 (Tight quantity restriction)**.** *Assume the inequalities in Proposition 7 are satisfied. Suppose $q^\star \in [\underline{q}, 1]$ and $b^\star = \underline{b}_S$. Then,*

(i) The saving equilibrium is always an equilibrium. The third party incurs zero losses.

(ii) There is a $\hat{q} < 1$ such that for all $q^\star > \hat{q}$, the borrowing equilibrium does not exist.

The propositions above show that a price floor policy has very different implications, depending on the quantity restriction that accompanies it. If the quantity restriction is loose, a generous price floor ends up incentivizing borrowing and generates losses for the third party. However, if the quantity restriction is tight enough, a generous price floor eliminates the sub-optimal borrowing equilibria, and no resources are lost by the third party on equilibrium. In fact, in the latter case, the third party never needs to purchase debt in equilibrium.

Recall that the multiplicity reflects the trade-off between saving for a better price versus the desire to borrow due to impatience. With a price floor absent a tight quantity restriction, the third party reduces the incentive to save. The saving equilibrium is supported by the gap between prices in the Safe Zone and princes in the Crisis Zone as well as the need to roll over bonds. A generous price floor in the Crisis Zone eliminates the price differential that incentivizes saving in equilibrium.

Rewarding the government for saving, or punishing them for borrowing, is a policy that can induce the saving equilibrium. A borrowing limit at the boundary of the Safe Zone, which is tighter than the endogenous limit, would be effective. However, such a policy raises the question of how to enforce the limit if the initial debt is beyond it. Third-party purchases conditional on fiscal austerity are reminiscent of policies pursued in the European debt crisis as well as many IMF programs. However, the events in Europe and elsewhere reflect the difficulties of enforcing explicit debt limits. Unfortunately, in the Eaton-Gersovitz framework studied in this paper, there is no effective policy that does not involve a similar type of off-equilibrium commitment to punish overborrowing.

Finally, note that a tight quantity restriction policy may not be effective if delayed too long. In particular, once $b > b^I$, the saving equilibrium is no longer distinguishable from the borrowing equilibrium, and thus policy interventions will fail to be effective once debt has reached sufficiently high levels. This highlights that interventions during debt crises may need to be quick to be successful, and policies that “kick the can down the road” may eventually fail. This same point about delay, although in a different environment, was emphasized by Lorenzoni and Werning (2013).

7 Multiplicity in a Quantitative Model

Our previous analysis emphasized transparency in order to identify and analyze the economic forces that generate inefficiencies and lead to multiplicity. Of course, this required several simplifying assumptions. In this section, we relax these assumptions and show that the insights of the theory extend to richer environments, such as those used in quantitative analyses.[21]

Towards this end, we use the state-of-the-art sovereign debt model of Chatterjee and Eyigungor (2012) – henceforth CE12 – which features discrete time, concave utility for the government, endowment risk, non-linear default punishment, as well as the possibility of re-accessing financial markets after default. In this section, we demonstrate that such an environment is prone to the multiplicity identified by our analytical framework.

Because we hew very closely to the benchmark CE12 specification, we relegate most details of the set up and its computation to Appendix E. We flag here the one change we make to the environment. CE12 calibrate their model to Argentina. They estimate an AR(1) endowment process and approximate this using a discrete Markov chain. We augment the endowment process by including a “rare disaster” state. This proves useful in computing the two equilibria. It also has empirical validity given the work of Barro and Ursa (2008), Barro (2011), and has recently

[21] Stangebye (2018) discusses the role of concavity of utility and re-entry dynamics in generating multiplicity in a similar environment.

been introduced in the sovereign debt context by Ayres et al. (2015), Rebelo et al. (2019), and Paluszynski (2019).

Specifically, the endowment y_t follows a discretized AR(1) process during "normal" times, but with constant probability π_{dis} switches to a disaster state y_{dis}. Once in the disaster state, it recovers with probability π_{rec}, at which point it resumes following the normal AR(1) process. Following Barro and Ursa (2008), we set π_{dis} to be 0.97%, and y_{dis} to be 0.20 log points below the mean of the normal AR(1) process. Barro and Ursa (2008) estimates the average length of a disaster to be 3.5 years, and hence we set $\pi_{rec} = 7.14\%$ in our quarterly model. We adjust the auto-correlation parameter and innovation variance underlying the normal AR(1) process to match the auto-correlation and volatility of GDP targeted by CE12.

The addition of the disaster state also involves choosing the fraction of endowment lost due to default in that state. We assume that 4.5% of the disaster endowment state is lost while in default status. We shall discuss that choice in detail below. For the remainder of the endowment process, we use the default cost specified in CE12.

Other than the enriched endowment state vector and the associated additional default cost, the remaining details and parameter values are identical to CE12. In particular, we set the benchmark expected maturity to 20 quarters. The quarterly risk-free interest is 1% and the government's quarterly discount factor is 0.954. The remainder of the parameters are reported in the appendix.

The model features (at least) two equilibria at CE12's calibrated expected maturity of 20 quarters. Mirroring our analytical model, one equilibrium is a "saving" equilibrium, in which an indebted government saves in order to attain a risk-free price. For the same parameterization, we compute a "borrowing" equilibrium, in which the ergodic distribution features high debt and recurrent default. The policy functions and associated price schedules are depicted in Appendix E Figures E.1 and E.2. The appendix also contains business cycle moments for both equilibria.

As in the analytical model, the quantitative saving equilibrium features an absorbing "Safe Zone." That is, there exists a threshold $\underline{b}_S$ such that if $b \leq \underline{b}_S$, the government never defaults and faces risk-free prices. As is standard in quantitative default models with non-contingent bonds, the government's incentive to default is greatest when the endowment is lowest. Thus, $\underline{b}_S$ is defined by the debt level at which the government is indifferent between default and repayment when the endowment is in the disaster state. In our benchmark calibration, the market value of $\underline{b}_S$ is 1.16, relative to an average (quarterly) endowment of 1.01 and a disaster endowment of 0.84. As in the analytical model, there is a region of debt levels above this threshold in which the government saves (conditional on positive endowment realizations and no default) in order to exit the Crisis Zone.

In the borrowing equilibrium, the government borrows into the Crisis Zone and eventually defaults. This equilibrium displays the familiar pattern from the quantitative literature in that starting from zero debt, the government leverages up and then eventually defaults with probability one. The ergodic mean of the face value of debt-to-GDP ratio is 0.96. For reference, CE12 achieve an average ratio of 0.70 in their benchmark simulation. The remaining ergodic moments of the borrowing equilibrium are also mostly in line with those of CE12.

Proposition 7 established that there is a non-trivial interval of maturity for which multiple equilibria can be supported. In the quantitative version, we successfully computed a saving equilibrium for maturities ranging from 1 to 33 quarters. The borrowing equilibrium can be computed for maturities as short as 9 quarters. Thus, Proposition 7 has quantitative bite – *there is a quantitatively significant range of maturities for which multiple equilibria exist.*

Proposition 7 also emphasized the role of the deadweight costs of default in generating a saving equilibrium. To map our default cost choice into a deadweight cost, recall that the the market value of debt at $\underline{b}_S$ is 1.16. At this debt level and in the disaster state, the government is indifferent between repayment and default, while the lenders lose 1.16 in expected present value. Hence, the market value of the threshold represents the surplus lost by default at $\underline{b}_S$ in state y_{dis}. This is equivalent to a loss of 1.17% of the expected present value of the government's endowment starting from the disaster state (which is 99.34).

Whether this is empirically reasonable is difficult to determine, as we need to compare the post-default endowment process to the counter-factual process if the government had not defaulted. One recent attempt to measure this is Hébert and Schreger (2017), which uses the behavior of Argentine equities on US exchanges around news events relating to litigation involving hold-out creditors. Their estimates suggest that an unanticipated default generates a decline in market value of equity of 45 percent. Thus, the one percent of the expected present value of GDP is not unreasonably large.

The standard approach in the quantitative literature is to indirectly calibrate the default costs in order to generate significant borrowing combined with frequent default. The outcome is typically a very small cost in the low-endowment states that encourages default, combined with a disproportionately larger cost in high-endowment states in order to sustain borrowing during booms. For example, if we use CE12's functional form and extend it to our disaster state, we compute a deadweight cost (that is, the market value of debt that makes the government indifferent between default or not) of 0.70, or sixty percent of our benchmark number. At this low cost, a saving equilibrium cannot be sustained even with one-period bonds. Hence, the competitive equilibrium will not support an absorbing Safe Zone at any maturity. The lowest cost for which we found both a saving and borrowing equilibrium at some maturity is 0.88.

These experiments reveal two important lessons for quantitative sovereign debt models. One is that multiplicity is possible in such models for a wide range of empirically relevant maturities, as long as default costs are not too small. The second is that the practice of calibrating nonlinear default costs in order to match debt and default frequencies in the data may naturally lead to environments in which an absorbing Safe Zone is not constrained efficient, and hence a saving equilibrium may be unlikely to exist. However, such low default costs are not directly tied to empirical evidence and this practice may provide an incomplete picture regarding the vulnerability to self-fulfilling dilution.

8 Relationship to Other Sources of Multiplicity

We conclude with some comments on the relationship of our analysis to alternative models of multiplicity. The core driving force behind the multiplicity is the feedback between the price schedule and the government's debt-issuance policy function. In particular, the incentives embedded in equilibrium prices to alter the outstanding stock of debt before legacy bonds mature. It is useful to contrast this mechanism with other environments that feature multiplicity.

The source of multiplicity studied above is distinct from the canonical Calvo (1988) multiplicity. A useful way to view the Calvo multiplicity is through the feasibility of debt trajectories. That is, the mechanical link via the budget set between low prices of bond issuances today and high debt burdens tomorrow. In the two-period model of Calvo (1988), today's bond prices (or the implied interest rate) determine the debt burden tomorrow, given the requirement to raise a certain amount of revenue in the initial period. Lorenzoni and Werning (2013) (henceforth, LW) explore how to extend this mechanism to a fully dynamic model.

LW emphasize that, in practice, governments have limited flexibility to alter fiscal policy at high frequencies. Most of their analysis therefore assumes the government follows a fiscal rule. However, they show that there is a natural counterpart to a fiscal rule in an environment with a fully optimizing government. In particular, they consider a government making optimal debt decisions subject to a minimum threshold for spending (which in their case arises naturally from the non-negativity of spending).

LW multiplicity turns on whether the government is *able* to reduce its debt when facing low bond prices. In particular, they consider a situation where the bond price may be low enough that it is not feasible for the government to reduce debt. As a result, the government optimally chooses to accumulate debt instead, justifying the low bond prices. Importantly, they show how such a feasibility constraint on expenditure can generate multiplicity even though it does not bind on the equilibrium path – once the governments decides to borrow, its expenditures are not constrained by the lower bound. But, as noted by LW, the potentially binding constraint is necessary to generate the multiplicity they study.

It is useful to highlight how the LW multiplicity differs from the one in our analysis. The equivalent of a minimum spending threshold in our environment is the consumption lower bound

$\underline{C}$. In our equilibria, we assume that this lower bound does not bind.[22] Moreover, even when facing the "borrowing" equilibrium prices, we implicitly assume that it is always *feasible* but not optimal for the government to save towards the Safe Zone.

This begs the question of whether the LW multiplicity arises if the consumption lower bound $\underline{C}$ is high enough. To answer this, let us now attempt to construct the LW multiplicity in our environment. First, consider parameters, including $\underline{C}$, such that the saving equilibrium exists. To construct an alternative LW "borrowing" equilibrium, suppose that there exists a $b_0 \in (\underline{b}_S, \overline{b}_B)$ such that reducing the debt is *not feasible at the borrowing equilibrium prices*, $\underline{q}$.[23] That is,

$$\underline{C} > y - (r + \delta(1 - \underline{q}))b_0. \tag{28}$$

This condition implies that, at a price of $\underline{q}$ and $b \geq b_0$, even if the government were to set its consumption to its lower bound, debt still strictly increases. Such a restriction is used by LW to sustain an alternative equilibrium for high enough debt levels: given that the government cannot reduce debt, it is now willing to follow the borrowing equilibrium prescription and accumulate debt until $\overline{b}_B$ (justifying the price of $\underline{q}$).

However, in our environment, such an equilibrium cannot exist. To see this, if equation (28) holds, Assumption 1 (iv) implies that $b_0 > \overline{b}_B$.[24] The reason is that if $\dot{b} \leq 0$ is not feasible at $\underline{q}$ and $b = b_0$, then $\dot{b} > 0$ for all $b > b_0$, including the upper-bound $\overline{b}_B$, violating the equilibrium conditions.[25]

Interestingly, while imposing a tight lower bound on consumption in our environment does not generate the equilibrium studied by LW, it can produce a roll-over crisis a la Cole and Kehoe (2000) (henceforth CK). Consider again a situation in which savings is an equilibrium. We ask the question of whether we can also sustain an equilibrium where the bond price switches from the savings equilibrium price to a price of zero within a subset of the domain $(\underline{b}_S, \overline{b}_S)$.

We conjecture the following "failed-auction" equilibrium: there exists a $b_0 \in (\underline{b}_S, \overline{b}_S)$ and an equilibrium price q such that $q(b) = 0$ for $b > b_0$, and $q(b) = q_S(b)$ otherwise, where q_S is the saving equilibrium price schedule. Note that this implies $V_S(b_0) > \underline{V}$.

[22] We introduced a lower bound on consumption only to have a well defined policy for the efficient allocation.

[23] Note that, the savings equilibrium is not necessarily affected by the lower bound. That is, it could be the case that reducing the debt remains feasible at high prices. A sufficient condition in our case is that $\hat{C}(b) = y - (r + \delta(1 - \hat{q}(b))b + \hat{q}(b)\dot{b} > \underline{C}$ for $b \in (\underline{b}_S, b_0)$, where $\dot{b}$ is given by equation (36) in the appendix.

[24] A remaining issue is what would happen if we were to drop Assumption 1 (iv) from our requirements. In that case, in the borrowing allocation, the maximal amount of debt, $\overline{b}_B$ that could be feasibly sustained in a borrowing equilibrium is such that $\underline{C} = y - (r + \delta(1 - \underline{q}))\overline{b}_B$. Inequality (28) still implies $b_0 > \overline{b}_B$.

[25] Differently from us, LW assume that there is a positive recovery rate upon default and impose an upper-bound on debt, at which point, a renegotiation between creditors and the government is automatically triggered. This upper-bound guarantees that debt does not grow without bound in the "bad" equilibrium, even if it cannot be kept stationary absent renegotiation.

First, suppose $\underline{C} < y-(r+\delta)b$ for $b \in (\underline{b}_S, \overline{b}_S)$. This implies that it is feasible for the government to reduce debt by paying coupons and maturing bonds. Now consider a b in the neighborhood above b_0. The government has the option to reduce debt to b_0 by paying off maturing bonds. The value of this strategy is arbitrarily close to $V_S(b_0)$ as $b \downarrow b_0$. As $V_S(b_0) > \underline{V}$, the government will not find it optimal to default in this neighborhood, invalidating zero as an equilibrium price. With no lower bound on consumption, the government can always act as its own lender of last resort in case of a rollover crisis, eliminating the CK multiplicity.

Alternatively, continuing the premise that the saving equilibrium exists, suppose $\underline{C}$ satisfies the constraint (28) for a $b_0 \in (\underline{b}_S, \overline{b}_S)$.[26] For $b > b_0$, there is no feasible option for the government other than default. This follows from the fact that (28) implies the government cannot pay off its coupon and maturing principal payments at $b = b_0$ when the price is $\underline{q} > 0$; hence, it cannot do so at $b > b_0$ when facing a zero price. Thus, the government must default, validating the zero price in equilibrium.[27]

In summary, the difference between the equilibria we study and those of Calvo, CK, and LW reflect different views with regards to fiscal policy. This paper emphasizes the *lack* of constraints on fiscal policy, bringing limited commitment to future fiscal paths to the fore of the analysis. Calvo, CK, and LW emphasize the *limitations* of fiscal policy when responding to low bond prices. Both views are complementary, and highlight the potential fragility of sovereign debt markets to changes in lender expectations.

9 Conclusion

This paper shows that debt dilution generates multiplicity in a standard sovereign debt framework. In particular, the extent of dilution in equilibrium depends on self-fulfilling expectations of future prices and future fiscal policy. A relatively impatient government, an intermediate debt maturity, and deadweight losses from default provide the conditions for multiplicity of equilibria. Importantly, these are common features of observed debt markets as well as the recent quantitative models proposed in the literature.

The framework presented above is designed for analytical clarity and thus involves some special assumptions. However, the mechanism at work is robust to including endowment fluctu-

[26]Note that this implies from Assumption 1 (iv) that $b_0 > \overline{b}_B$.

[27]In a discrete time environment, with one period bonds, the difference between the uniqueness in the Eaton-Gerstovitz model (Auclert and Rognlie, 2016, Aguiar and Amador, 2019) and the multiplicity in the Cole and Kehoe (1996) model is explained in terms of the timing within a period. In Eaton-Gersovitz, the government commits to default or not *before* issuing the new bonds. In Cole-Kehoe, the government first issues the new bonds, then decides to default or not. In continuous time, with long duration bonds, this within-period distinction is not relevant. This section highlights that, if there is a consumption lower bound, roll-over crises can indeed co-exist with the type of multiplicity we identify in the continuous-time Eaton-Gersovitz framework.

ations and risk aversion, which, while bringing the model closer to empirical debt markets, does not eliminate the self-fulfilling debt dilution identified in the tractable model. One can easily construct simple numerical examples of multiplicity with these elements. The quantitative model analyzed by Stangebye (2015) also appears to be driven by a mechanism similar to that studied in this paper. Indeed, the fact that the Eaton-Gersovitz model is vulnerable to dilution is at the heart of the recent quantitative literature that attempts to match empirical sovereign debt crises. We show that the same force leads to indeterminacy. The fact that multiplicity stems from the incentives to dilute places novel restrictions on effective third-party interventions.

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Appendix A Closed-Form Expressions and Derivations

In this appendix, we provide closed-form expressions for the solutions to the planning problem as well as equilibrium objects. We also include some notes on the underlying derivations.

A.1 The Efficient Borrowing Allocation

The conjectured policy function for consumption in the borrowing allocation is given in (5). As $\underline{V}$ is a stationary point, we immediately have:

$$P_B^\star(\underline{V}) = \frac{y - C^\star(\underline{V})}{r+\lambda}.$$

Given this boundary condition and the consumption policy function, we solve the ODE (P) to obtain:

(i) For $v \in [\underline{V}, \overline{V}]$:

$$P_B^\star(v) \equiv \frac{1}{r+\lambda}\left[y - \overline{C} + \frac{(\overline{C} + \lambda\overline{V} - (\rho+\lambda)v)^{\frac{r+\lambda}{\rho+\lambda}}}{(\overline{C} + \lambda\overline{V} - (\rho+\lambda)\underline{V})^{\frac{r-\rho}{\rho+\lambda}}}\right].$$

(ii) For $v \in (\overline{V}, V_{max}]$:

$$P_B^\star(v) \equiv \frac{1}{r}\left[y - \overline{C} + (\overline{C} - y + rP_B^\star(\overline{V}))\frac{(\overline{C} - \rho v)^{\frac{r}{\rho}}}{(\overline{C} - \rho\overline{V})^{\frac{r}{\rho}}}\right].$$

A.2 Efficient Saving

For the efficient saving allocation, the conjecture is that the Safe Zone is an absorbing state. In particular, $\overline{V}$ is a stationary point, which pins down $P_S^\star(\overline{V}) = (y - \rho\overline{V})/r$. Given this boundary condition and the policy function (9), we solve (P) for the Safe Zone to obtain:

$$P_S^\star(v) \equiv \frac{1}{r}\left[y - \overline{C} + \left(\overline{C} - \rho\overline{V}\right)^{\frac{\rho-r}{\rho}}\left(\overline{C} - \rho v\right)^{\frac{r}{\rho}}\right] \text{ for } v \in [\overline{V}, V_{max}]. \tag{29}$$

For the saving region of the Crisis Zone, consumption is at its lower bound, $\underline{C}$. Solving (P) for $v \in [\underline{V}, \overline{V})$, using $P_S^\star(\overline{V})$ from above as a boundary condition, we obtain the planner's value under saving:

$$\hat{P}(v) \equiv \frac{1}{r+\lambda}\left[y - \underline{C} + (\underline{C} - y + (r+\lambda)P_S^\star(\overline{V}))\left(\frac{\underline{C} + \lambda\overline{V} - (\rho+\lambda)v}{\underline{C} - \rho\overline{V}}\right)^{\frac{r+\lambda}{\rho+\lambda}}\right]. \tag{30}$$

Per equation (12), $P_S^\star(v)$ in the Crisis Zone is the maximum of $\hat{P}$ and $P_B^\star$. Straightforward differentiation indicate that $\hat{P}$ and $P_B^\star$ cross at most once.

A.3 The Borrowing Equilibrium

In the Crisis Zone $(\underline{b}_B, \overline{b}_B]$, $q_B(b) = \underline{q}$. Turning to the Safe Zone, recall that the conjectured consumption policy function in the borrowing equilibrium is the same as the planner's borrowing policy, (5). With this policy and the boundary $q_B(\overline{b}_B) = \underline{q}$, the solution to (20) is defined implicitly by:

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}}\right)^{\frac{r}{r+\delta}} = \frac{\overline{C} - y + rq_B(b)b}{\overline{C} - y + r\underline{q}\underline{b}_B}. \tag{31}$$

For each $b \in [0, \underline{b}_B)$, there is a unique solution for $q_B(b) \in [\underline{q}, 1]$. Recall that for $b < 0$, we have $q_B(b) = 1$ regardless of the government's policies.[28]

The government's value in the borrowing equilibrium is obtained by inverting $P_B^\star$. Specifically:

$$V_B(b) = \begin{cases} \frac{1}{\rho}\left(\overline{C} - (\overline{C} - \rho\overline{V})\left(\frac{\overline{C}-y+rq_B(b)b}{\overline{C}-y+r\underline{q}\underline{b}_B}\right)^{\frac{\rho}{r}}\right) & \text{for } b \in [-\overline{a}, \underline{b}_B] \\ \frac{1}{\rho+\lambda}\left(\overline{C} + \lambda\overline{V} - \frac{\left(\overline{C}-y+(r+\lambda)\underline{q}b\right)^{\frac{\rho+\lambda}{r+\lambda}}}{\left(\overline{C}-y+(r+\lambda)\underline{q}\overline{b}_B\right)^{\frac{\rho-r}{r+\lambda}}}\right) & \text{for } b \in (\underline{b}_B, \overline{b}_B], \end{cases} \tag{32}$$

where $\overline{a} \equiv (\overline{C} - y)/r$ is the maximal net inflows that can be consumed by the government.

A.4 The Saving Equilibrium

The saving equilibrium objects in the Safe Zone is straightforward: because it is an absorbing region, there is no risk of default starting from $b \leq \underline{b}_S$. Hence, the price is one, $q_S(b) = 1$ and the values and consumption are equivalent to their efficient counterparts. That is, inverting $P_S^\star$ we obtain:

$$V_S(b) = \rho^{-1}\left(\overline{C} - (\overline{C} - \rho\overline{V})\left(\frac{\overline{C} - y + rb}{\overline{C} - y + r\underline{b}_S}\right)^{\frac{\rho}{r}}\right) \text{ for } b \in [-\overline{a}, \underline{b}_S] \tag{33}$$

where $\underline{b}_S \equiv (y - \rho\overline{V})/r$. The consumption policy is $\overline{C}$ for $b < \underline{b}_S$, and $\rho\overline{V}$ at $\underline{b}_S$.

Turning to the Crisis Zone, we begin with the saving region. Let $\{\hat{V}, \hat{C}, \hat{q}\}$ denote the conjectured equilibrium objects in the saving region of the Crisis Zone. In the saving region, we have to deviate from the prescription of the efficient allocation. The reason is that the efficient savings policy, which sets consumption at its lower bound $\underline{C}$, cannot be sustained in a competitive equilibrium. That is, the efficient savings rate is not privately optimal in an equilibrium with long-term bonds.

We conjecture instead that the government saves by consuming at an interior optimum.[29] When consumption is interior, the linearity of the government's objective function in (17) implies that it is indifferent across alternative consumption choices, including the consumption level that sets $\dot{b} = 0$. Hence, the government must be indifferent between the equilibrium consumption strategy and its associated stationary

[28] Note that there may be a discontinuity in q_B at $b = 0$. Recall that at points of discontinuity, we impose that debt buybacks occur at a price of one in the neighborhood around a discontinuity. This restriction eliminates the technical complication of the government attempting to issue debt at one price and near-simultaneously repurchasing at a lower price in an attempt to exploit this discontinuity. The restriction we impose ensures that the choice set is convex despite the discontinuity in price, and hence the government has no motive to "mix" by moving consumption back and forth while keeping debt at the point of discontinuity.

[29] Throughout the following analysis, we assume $\underline{C}$ is sufficiently low that an interior consumption choice is feasible.

value:[30]

$$\hat{V}(b) \equiv \frac{y - [r + \delta(1 - \hat{q}(b))]b + \lambda\overline{V}}{\rho + \lambda}. \tag{34}$$

From the first-order condition in (17), interior consumption requires $\hat{V}'(b) = -\hat{q}(b)$. Using this, differentiating (34), and solving the resulting ODE with $\hat{q}(\underline{b}_S) = 1$ as a boundary condition yields

$$\hat{q}(b) \equiv \frac{r + \delta + \left(\frac{b}{\underline{b}_S}\right)^{-\frac{\rho+\lambda+\delta}{\delta}} (\lambda + \rho - r)}{\rho + \lambda + \delta}. \tag{35}$$

The lenders' break-even condition (19) requires $\hat{q}'(b)\dot{b} = (r + \delta + \lambda)\hat{q}(b) - (r + \delta)$. Hence, we can solve for the conjectured debt dynamics:

$$\dot{b} = -\delta b \left(\frac{\hat{q}(b) - \underline{q}}{\hat{q}(b) - \underline{q} + \frac{(\rho - r)\hat{q}(b)}{r+\delta+\lambda}} \right) \equiv f(b). \tag{36}$$

Using (16), we obtain the associated consumption:

$$\hat{C}(b) \equiv y - [r + \delta(1 - \hat{q}(b))]b + \hat{q}(b)f(b). \tag{37}$$

The borrowing region of the Crisis Zone is also an absorbing state and corresponds to the equilibrium discussed in the previous subsection. Note that in this region, the price is $\underline{q}$. In the Crisis Zone, $V_S(b) = \max\langle \hat{V}(b), V_B(b) \rangle$. As before, b^I is the intersection point of these two alternatives. If no such $b^I \in [\underline{b}_S, \overline{b}_B]$ exists, we set it to $\overline{b}_S$. The value of $\overline{b}_S$ is such that $V_S(\overline{b}_S) = \underline{V}$, and we define $\boldsymbol{B}_S \equiv [-\overline{a}, \overline{b}_S]$.[31]

The saving equilibrium value in the Crisis Zone is therefore:

$$V_S(b) \equiv \begin{cases} \hat{V}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ V_B(b) & \text{for } b \in (b^I, \overline{b}_S]; \end{cases} \tag{38}$$

and the consumption policy is

$$C_S(b) \equiv \begin{cases} \hat{C}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ C_B(b) & \text{for } b \in (b^I, \overline{b}_S]. \end{cases} \tag{39}$$

[30] The fact that the government's value is equal to the stationary value while consumption is interior is discussed in Tourre (2017) and DeMarzo et al. (2018). The authors give an interpretation of a durable monopolist in the spirit of the Coase conjecture.

[31] If $b^I < \overline{b}_B$, then q_S is discontinuous at b^I, which is the case depicted in Figure 3. As previously discussed when stating the government's problem, and echoed in footnote 28, we rule out the government issuing at $q_S(b^I)$ and then immediately repurchasing at $\lim_{b' \downarrow b^I} q_S(b') < q_S(b^I)$ in an attempt to set $\dot{b} = 0$ by alternating between issuing and repurchasing. Let us also note that the multiplicity result we obtain later on does not hinge on this particular issue: it is possible to obtain parameter values such that $b^I = \overline{b}_S$ and for which multiple equilibria coexist.

The equilibrium price schedule is:

$$q_S(b) \equiv \begin{cases} 1 & \text{for } b \in [-\overline{a}, \underline{b}_S] \\ \hat{q}(b) & \text{for } b \in [\underline{b}_S, b^I] \\ \underline{q} & \text{for } b \in (b^I, \overline{b}_S]; \end{cases} \tag{40}$$

Appendix B Additional Results

In this appendix, we state state four results. The first two allows us to provide a characterization of a solution to the planner's problem. The next two are the same results for the government's problem in a competitive equilibrium.

The value function $P^\star(v)$ has the following standard properties:

Lemma B.1. *The solution to the planner's problem, $P^\star(v)$, is bounded and Lipschitz continuous.*

Proof. The proof is in Appendix C. □

Lemma B.1 states that $P^\star$ is bounded and Lipschitz continuous, and hence differentiable almost everywhere. However, there may be isolated points of non-differentiability. At such points, $P^\star$ satisfies (P) in the viscosity sense. In particular:

Proposition B.1. *Suppose a bounded, Lipschitz continuous function $p(v)$ with domain $\mathbb{V}$ has the following properties:*

(i) p satisfies (P) at all points of differentiability;

(ii) If $\lim_{v\uparrow\overline{V}} p'(v) > \lim_{v\downarrow\overline{V}} p'(v)$ and $\lim_{v\uparrow\overline{V}} p'(v) \geq -1$, then $p(\overline{V}) = (y - \rho\overline{V})/r$;

(iii) At a point of non-differentiability $\tilde{v} \neq \overline{V}$, we have $\lim_{v\uparrow\tilde{v}} p'(v) < \lim_{v\downarrow\tilde{v}} p'(v)$;

(iv) If $p'(\underline{V}) < -1$, then $p(\underline{V}) = (y - \rho\underline{V} + \lambda(\overline{V} - \underline{V}))/(r + \lambda)$;[32] and

(v) $p'(V_{max}) \leq -1$;

then $p(v) = P^\star(v)$.

Proof. The proof is in the Online Appendix. □

The first condition of the proposition ensures that the candidate value function satisfies the HJB wherever it is smooth. The second condition concerns the case when $\overline{V}$ is a locally stable stationary point; this will be relevant when we consider an efficient "saving allocation" defined below. The third condition states that any other point of non-differentiability has a "convex" kink. The final two conditions are sufficient to ensure that v remains in $\mathbb{V}$.

The counterpart to Lemma B.1 for the equilibrium value function is:

Lemma B.2. *In any competitive equilibrium such that $q(b) \in [\underline{q}, 1]$ for $b \in \boldsymbol{B} = [-\overline{a}, \overline{b}]$, V is bounded, strictly decreasing, and Lipschitz continuous on $\boldsymbol{B}$.*

Proof. The proof is in Appendix C. □

[32]For the endpoints of $\mathbb{V}$, we interpret $p'(\underline{V}) \equiv \lim_{v\downarrow\underline{V}} p'(v)$ and $p'(V_{max}) \equiv \lim_{v\uparrow V_{max}} p'(v)$.

The counterpart to Proposition B.1 for the government's equilibrium problem (17) is:

Proposition B.2. *Consider the government's problem given a compact debt domain $\boldsymbol{B}$ and a price schedule $q : \boldsymbol{B} \to [\underline{q}, 1]$ that has a (bounded) derivative at almost all points in $\boldsymbol{B}$. If a strictly decreasing, Lipschitz continuous function $v : \boldsymbol{B} \to [\underline{V}, \overline{C}/\rho]$ has the following properties:*

(i) v satisfies (17) at all points of differentiability;

(ii) If $\lim_{b\uparrow\underline{b}} v'(b) > \lim_{b\downarrow\underline{b}} v'(b)$, then $\rho v(\underline{b}) = \rho\overline{V} = y - [r + \delta(1 - q(\underline{b}))]\underline{b}$;

(iii) At a point of non-differentiability $\tilde{b} \neq \underline{b}$, we have $\lim_{b\uparrow\tilde{b}} v'(b) < \lim_{b\downarrow\tilde{b}} v'(b)$;

(iv) $\rho v(-\overline{a}) = \overline{C}$; and

(v) $(\rho + \lambda)v(\overline{b}) = y - [r + \delta(1 - q(\overline{b}))]\overline{b} + \lambda\overline{V}$;

then $v(b) = V(b)$ is the government's value function.

Proof. The proof is in the Online Appendix. □

The conditions listed in the proposition are similar to those from Proposition B.1. Namely, that the value function satisfies the HJB equation with equality wherever smooth; there may be a local attractor that corresponds to $\underline{b}$ if the government saves; other points of non-differentiability have convex kinks; and the endpoints of the domain deliver the value of holding debt constant.[33]

Appendix C Proofs

This appendix contains all proofs except those for Propositions B.1 and B.2, which are presented in the Online Appendix, along with a discussion of viscosity solutions more generally.

C.1 Proof of Lemma 1

Proof. To generate a contradiction, suppose there is an efficient allocation $\{\boldsymbol{c}, T\}$, with $T < \infty$. Note from (1) we have $V(T, \boldsymbol{c}) = \underline{V}$. To see this, suppose instead that $V(T, \boldsymbol{c}) > \underline{V}$; that is,

$$\begin{aligned} V(T, \boldsymbol{c}) &= \sup_{T' \geq t} \int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s)\mathrm{d}s + e^{-(\rho+\lambda)(T'-T)}\underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, \boldsymbol{c}), \overline{V}\rangle \mathrm{d}s \\ &> \underline{V}. \end{aligned}$$

Hence, there exists a $T' > T$ such that

$$\int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s)\mathrm{d}s + e^{-(\rho+\lambda)(T'-T)}\underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, \boldsymbol{c}), \overline{V}\rangle \mathrm{d}s > \underline{V}.$$

[33] Condition (v), at $\overline{b}$, is stronger than necessary, as the key requirement is that $\dot{b} \leq 0$ at the upper bound on debt; however, in the equilibria described below, the stronger condition is always satisfied.

This implies at time $t < T$,

$$\int_t^T e^{-(\rho+\lambda)(s-t)} c(s)\mathrm{d}s + e^{-(\rho+\lambda)(T-t)}\underline{V} + \lambda \int_t^T e^{-(\rho+\lambda)(s-t)} \max\langle V(s,\boldsymbol{c}), \overline{V}\rangle \mathrm{d}s <$$
$$\int_t^{T'} e^{-(\rho+\lambda)(s-t)} c(s)\mathrm{d}s + e^{-(\rho+\lambda)(T'-t)}\underline{V} + \lambda \int_t^{T'} e^{-(\rho+\lambda)(s-t)} \max\langle V(s,\boldsymbol{c}), \overline{V}\rangle \mathrm{d}s.$$

Hence, T was never a sup of the original problem. This establishes that $V(T, \boldsymbol{c}) = \underline{V}$.

Now consider an alternative allocation $(\tilde{\boldsymbol{c}}, \infty)$. The alternative consumption allocation equals $\boldsymbol{c}$ for $t < T$, but differs for $t \geq T$. We choose $\tilde{c}(t) = (\rho + \lambda)\underline{V} - \lambda\overline{V} < y$ for $t \geq T$ so that for all $t \geq T$:

$$\begin{aligned} V(t, \tilde{\boldsymbol{c}}) &= \frac{\tilde{c}(t) + \lambda\overline{V}}{\rho + \lambda} \\ &= \frac{(\rho + \lambda)\underline{V} - \lambda\overline{V} + \lambda\overline{V}}{\rho + \lambda} \\ &= \underline{V}. \end{aligned}$$

Thus, $V(0; \boldsymbol{c}) = V(0; \tilde{\boldsymbol{c}})$. Moreover, the alternative allocation delivers strictly more than zero to the lender in expectation for $t \geq T$ as $\tilde{c}(t) < y$. As the government is indifferent and the lender receives strictly more in expected present value, the original allocation is not efficient. □

C.2 Proof of Lemma B.1

Proof. Lemma 1 allows us to set $T = \infty$ in the planning problem (3) to obtain

$$P^\star(v) = \sup_{\boldsymbol{c}\in\boldsymbol{C},} \int_0^\infty e^{-\int_0^t r + \mathbb{1}_{[v(s)<\overline{V}]}\lambda ds}[y - c(t)]\mathrm{d}t \tag{41}$$

$$\text{subject to} \begin{cases} v(0) &= v \\ \dot{v}(t) &= -c(t) + \rho v(t) - \mathbb{1}_{[v(t)<\overline{V}]}\lambda \left[\overline{V} - v(t)\right], \end{cases}$$

defined on the domain $v \in \mathbb{V}$. $P^\star$ is bounded above by $(y - \underline{C})/r$ and below by $(y - \overline{C})/r$. To see that $P^\star$ is Lipschitz continuous in v, consider $v_1, v_2 \in \mathbb{V}$, with $v_2 > v_1$. A feasible strategy starting from $v(0) = v_2$ is to set consumption to $\overline{C}$ until $v(t) = v_1$. Let Δ denote the time $v(t)$ reaches v_1. Suppose $v(t) > \overline{V}$ for $t \in [0, \Delta_1)$ and $v(t) < \overline{V}$ for $t \in (\Delta_1, \Delta]$. Let $\Delta_2 = \Delta - \Delta_1$. If $v_2 < \overline{V}$, then $\Delta_1 = 0$ and if $v_1 > \overline{V}$, then $\Delta_2 = 0$. The dynamics of $v(t)$ imply

$$\begin{aligned} e^{-\rho\Delta_1} &= \frac{\overline{C} - \rho\max\{v_2, \overline{V}\}}{\overline{C} - \rho\max\{v_1, \overline{V}\}} \\ e^{-(\rho+\lambda)\Delta_2} &= \frac{\overline{C} + \lambda\overline{V} - (\rho + \lambda)\min\{v_2, \overline{V}\}}{\overline{C} + \lambda\overline{V} - (\rho + \lambda)\min\{v_1, \overline{V}\}}. \end{aligned}$$

Using this, one can show that

$$1 - e^{-\rho\Delta_1 - (\rho+\lambda)\Delta_2} \leq L|v_2 - v_1|, \tag{42}$$

with $L \equiv (\rho + \lambda)/(\overline{C} - \rho V_{max}) \in (0, \infty)$.

As this is a feasible strategy for v_2, integrating the objective function, we obtain

$$P^\star(v_2) \geq (y - \overline{C}) \left(\frac{1 - e^{-r\Delta_1}}{r} + \frac{e^{-r\Delta_1}\left(1 - e^{-(r+\lambda)\Delta_2}\right)}{r+\lambda} \right) + e^{-r\Delta_1 - (r+\lambda)\Delta_2} P^\star(v_1).$$

As $y < \overline{C}$, we have

$$P^\star(v_2) \geq (y - \overline{C}) \left(\frac{1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}}{r} \right) + e^{-r\Delta_1 - (r+\lambda)\Delta_2} P^\star(v_1),$$

which implies

$$P^\star(v_1) - P^\star(v_2) \leq \left(\frac{\overline{C} - y}{r} + P^\star(v_1) \right) \left(1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}\right).$$

As $P^\star(v_1) \leq (y - \underline{C})/r$, we have

$$P^\star(v_1) - P^\star(v_2) \leq \left(\frac{\overline{C} - \underline{C}}{r} \right) \left(1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}\right).$$

As $\overline{C} > \underline{C}$ and $\rho \geq r$, this implies

$$\begin{aligned} P^\star(v_1) - P^\star(v_2) &\leq \left(\frac{\overline{C} - \underline{C}}{r} \right) \left(1 - e^{-\rho\Delta_1 - (\rho+\lambda)\Delta_2}\right) \\ &\leq \left(\frac{\overline{C} - \underline{C}}{r} \right) L|v_2 - v_1|, \end{aligned}$$

where the second line uses (42). As $v_1 < v_2$, and hence $P^\star(v_1) \geq P^\star(v_2)$ as $P^\star$ is the efficient frontier, we have

$$|P^\star(v_1) - P^\star(v_2)| \leq K|v_2 - v_1|,$$

where

$$K \equiv \left(\frac{\overline{C} - \underline{C}}{r} \right) L = \left(\frac{\overline{C} - \underline{C}}{\overline{C} - \rho V_{max}} \right) \left(\frac{\rho + \lambda}{r} \right).$$

Hence, $P^\star$ is Lipschitz continuous with coefficient $K \in (0, \infty)$. □

C.3 Proof of Proposition 1

Proof. We need to check the conditions of Proposition B.1. Note that $P^\star_B$ is bounded, Lipschitz continuous, and differentiable everywhere except $\overline{V}$, where $\lim_{v \uparrow \overline{V}} P^{\star\prime}(v) < \lim_{v \downarrow \overline{V}} P^{\star\prime}(v)$. This inequality implies that condition (ii) in the proposition is irrelevant. Condition (iii) of Proposition B.1 is satisfied trivially. Condition (iv) is satisfied by construction.

At points of differentiability, the first-order condition for consumption requires $P^{\star\prime}_B(v) \leq -1$ for $c = \overline{C}$ to be optimal. Starting with $v \in [\underline{V}, \overline{V})$, differentiating the candidate function yields $P^{\star\prime}_B(v) \leq -1$. Hence $\overline{C}$

is optimal, and $P_B^\star$ satisfies the HJB on this domain. Turning to $v > \overline{V}$, note that $P_B^\star(v)$ is concave on this domain. Thus, if $\lim_{v\downarrow\overline{V}} P_B^{\star\prime}(v) \leq -1$, then $P_B^{\star\prime}(v) \leq -1$ for $v \in (\overline{V}, V_{max}]$. We have

$$\lim_{v\downarrow\overline{V}} P_B^{\star\prime}(v) = -\frac{\overline{C} - y + rP_B^\star(\overline{V})}{\overline{C} - \rho\overline{V}}.$$

This quantity is less than -1 when $rP_B^\star(\overline{V}) \geq y - \rho\overline{V}$. This is the condition stated in the proposition. This condition is necessary and sufficient for $P_B^\star$ to satisfy the HJB on $(\overline{V}, V_{max})$. Moreover, it is sufficient to ensure that condition (v) of Proposition B.1 is satisfied. □

C.4 Proof of Proposition 2

Proof. The proposed solution $P_S^\star$ is differentiable everywhere save $\overline{V}$ and v^I. At $\overline{V}$ we have $\lim_{v\uparrow\overline{V}} P_S^{\star\prime}(v) \geq -1 \geq \lim_{v\downarrow\overline{V}} P_S^{\star\prime}$. Hence, condition (ii) of Proposition B.1 is relevant and is satisfied by the candidate value function. $P_S^\star$ satisfies condition (iii) at v^I as it features a convex kink by construction. Condition (iv) is also satisfied by construction.

On the domain $v \in (\overline{V}, V_{max}]$, we have $P_S^{\star\prime}(v) \leq -1$, and hence $P_S^\star$ satisfies the HJB as well as condition (v) of Proposition B.1.

Turning to $v < \overline{V}$, we now show that $P_S^\star(\overline{V}) \geq P_B^\star(\overline{V})$ is necessary and sufficient for $P_S^\star$ to satisfy the conditions of Proposition B.1.

For sufficiency, suppose that $P_S^\star(\overline{V}) \geq P_B^\star(\overline{V})$. Let $X \equiv \{v \in [\underline{V}, \overline{V}) | P_S^\star(v) \geq P_B^\star(v)\} = [\max\{v^I, \underline{V}\}, \overline{V})$. On the domain X, $P_S^\star(v) = \hat{P}(v)$. One can show that $\hat{P}'(v) \geq -1$ if and only if $\hat{P}(v) \geq (y - (\rho + \lambda)v + \lambda\overline{V})/(r + \lambda)$. As the latter term is the value associated with setting $\dot{v} = 0$, the inequality is satisfied as $\hat{P}(v) \geq P_B^\star(v) \geq (y - (\rho + \lambda)v + \lambda\overline{V})/(r + \lambda)$. Hence $c = \underline{C}$ is optimal on X, and the HJB is satisfied. If $\hat{P}(\underline{V}) \geq P_B^\star(\underline{V})$, then $X = [\underline{V}, \overline{V})$, and hence the HJB is satisfied on the whole domain $\mathbb{V}$. If instead there exists $v^I > \underline{V}$, then the HJB is satisfied for $v < v^I$ from Proposition 1.

For necessity, suppose instead that $P_S^\star(\overline{V}) < P_B^\star(\overline{V})$. Comparison of the slopes implies that as long as $P_S^\star(v) < P_B^\star(v)$ for $v \in [\underline{V}, \overline{V})$, then $P_S^{\star\prime}(v) < P_B^{\star\prime}(v)$, and the two lines will never cross. Moreover, $P_B^{\star\prime}(v) \leq -1$, and hence $P_S^{\star\prime}(v) < -1$. This implies that $c = \underline{C}$ is strictly sub-optimal and the HJB is violated. □

C.5 Proof of Lemma B.2

Proof. The boundedness of V follows directly from $\overline{C}/\rho \geq V(b) \geq \underline{V}$ for any $b \in \boldsymbol{B}$.

To see that V is strictly decreasing, suppose $b_1 > b_2$ for $b_1, b_2 \in \boldsymbol{B}$. If $b_2 = -\overline{a} \equiv (y - \overline{C})/\rho$, then $V(b_2) = \overline{C}/\rho > V(b_1)$, where the latter inequality follows from the budget set at $b_1 > b_2$. Now consider the following policy starting from $b_2 \in (-\overline{a}, b_1)$: Set $c = \overline{C}$ until $b(t) = b_1$. As

$$\dot{b}(t) = \frac{c + (r + \delta)b(t) - y}{q(b(t))} - \delta b,$$

and $\overline{C} > y - rb \geq y - [r + \delta(1 - q(b))]b$ for $b \geq b_2$, we have $\dot{b}(t) > 0$. Let $\tilde{t} \in (0, \infty)$ denote when $b(t) = b_1$. As it is feasible for the government to follow this policy and not default while doing so, we have

$$V(b_2) \geq \int_0^{\tilde{t}} \overline{C} dt + e^{-\rho\tilde{t}} V(b_1) = \left(1 - e^{\rho\tilde{t}}\right) \frac{\overline{C}}{\rho} + e^{-\rho\tilde{t}} V(b_1).$$

Subtracting $V(b_1)$ from both sides yields:

$$V(b_2) - V(b_1) \geq \left(1 - e^{\rho\tilde{t}}\right)\left(\frac{\overline{C}}{\rho} - V(b_1)\right) > 0.$$

For continuity, we proceed in a similar fashion. Starting from b_1, consider the policy of setting $c = \underline{C}$ until $b(t) = b_2$. Let t^* denote the time where $b(t) = b_2$. Given that $\underline{C} < y - (r+\delta)\overline{b} \leq y - (r+\delta)b(t)$ and $q(b(t)) \in [\underline{q}, 1]$, $t^* < \infty$. Moreover, the same statements imply that

$$b_2 - b_1 \geq \int_0^{t^*} \left(\underline{C} + rb(t) - y\right) dt \geq \int_0^{t^*} \left(\underline{C} + r\overline{b} - y\right) dt = \left(\underline{C} + r\overline{b} - y\right) t^*,$$

where the first inequality follows from $q(b) \leq 1$.

The above implies that $t^* \geq L|b_1 - b_2|$, with $L \equiv \left(y - r\overline{b} - \underline{C}\right)^{-1} \in (0, \infty)$.

As this is a feasible strategy, we have

$$V(b_1) \geq \int_0^{t^*} e^{-\rho t}\underline{C} dt + e^{-\rho t^*} V(b_2) = (1 - e^{-\rho t^*})\frac{\underline{C}}{\rho} + e^{-\rho t^*} V(b_2),$$

where the inequality in the first line also reflects that the right-hand side is the value assuming the government never defaults, which is weakly below the optimal default policy. Subtracting $V(b_2)$ from both sides and rearranging, we have

$$V(b_2) - V(b_1) \leq (1 - e^{-\rho t^*})\left(V(b_2) - \frac{\underline{C}}{\rho}\right).$$

Using the fact that $\overline{C}/\rho > V(b) \geq \underline{V} > \underline{C}/\rho$ and $1 - e^{-\rho t^*} \leq t^*$, we have

$$0 < V(b_2) - V(b_1) \leq t^*\left(V(b_2) - \frac{\underline{C}}{\rho}\right) \leq L\left(\frac{\overline{C} - \underline{C}}{\rho}\right)|b_1 - b_2|.$$

Hence, $|V(b_2) - V(b_1)| \leq K|b_2 - b_1|$ with $K \equiv L\left(\overline{C} - \underline{C}\right)/\rho \in (0, \infty)$. □

C.6 Proof of Proposition 3

Proof. By construction, the price schedule q_B is consistent with the lenders' break-even condition, given the conjectured government policy. The remaining step is to verify if and when the government's policy is optimal given the conjectured q_B. Hence, to prove the proposition, we need to establish that V_B satisfies the conditions of Proposition B.2 if and only if (22) holds.

For $\overline{C}$ to be optimal for all $b < \overline{b}_B$, the first-order condition for the HJB requires $1 + V_B'(b)/q_B(b) \geq 0$ wherever $V_B'(b)$ exists. Thus, if $V_B'(b) \geq -q_B(b)$, then $c = \overline{C}$ is optimal. Recalling that V_B was constructed by assuming that the Hamiltonian is maximized at $c = \overline{C}$, then $V_B'(b) \geq -q_B(b)$ is both necessary and sufficient to verify that the HJB is satisfied at points of differentiability.

We proceed to show that (22) is equivalent to $V_B'(b) \geq -q_B(b)$ at points of differentiability.

For $b < 0$, we have

$$\rho V_B(b) = \overline{C} - (\overline{C} - \rho V_B(0)) \left(\frac{\overline{C} + rb - y}{\overline{C} - y} \right)^{\frac{\rho}{r}}.$$

Note that V_B is concave on this domain. For $\overline{C}$ to be optimal, it is therefore sufficient that $\lim_{b \uparrow 0} V_B'(b) \geq -1$. This will be true if and only if $\rho V_B(0) \geq y$. Hence, the condition in equation (22) evaluated at $b = 0$ is necessary and sufficient for the HJB to hold for $b \in (-\overline{a}, 0)$. For $b = -\overline{a} = (y - \overline{C})/\rho$, we have $V_B(-\overline{a}) = \overline{C}/\rho$, which is condition (iv) in Proposition B.2.

For $b \in (0, \underline{b}_B]$, from the lenders' break-even condition, in the Safe Zone, we have $(r + \delta) q_B(b) = q_B'(b)\dot{b} = q_B'(b) \left(\overline{C} + [r + \delta(1 - q_B(b))]b - y \right)$. Differentiating V_B in (32) and using this expression to substitute for $q_B'(b)$, we have for $b \in (0, \underline{b}_B]$

$$V_B'(b) = -q_B(b) \left(\frac{\overline{C} - \rho V_B(b)}{\overline{C} - [r + \delta(1 - q_B(b))]b - y} \right).$$

Hence, for $b \in (0, \underline{b}_B]$, the HJB is satisfied if and only if $\rho V_B(b) \geq y - [r + \delta(1 - q_B(b))]b$, which is the condition in equation (22).

For $b \in (\underline{b}_B, \overline{b}_B]$, we have $q_B(b) = \underline{q}$ and

$$(\rho + \lambda) V_B(b) = \overline{C} + \lambda \overline{V} - \left(\overline{C} + \lambda \overline{V} - (\rho + \lambda) \underline{V} \right) \left(\frac{\overline{C} - y + (r + \lambda) \underline{q} b}{\overline{C} - y + (r + \lambda) \underline{q} \overline{b}_B} \right)^{\frac{\rho + \lambda}{r + \lambda}}.$$

Note that $V_B(b)$ is concave in b, hence we need to check the condition at $b \to \overline{b}_B$. We have for $b \in (\underline{b}_B, \overline{b}_B]$

$$V_B'(b) \geq -\frac{\overline{C} + \lambda \overline{V} - (\rho + \lambda) \underline{V}}{\overline{C} - y + (r + \lambda) \underline{q} \overline{b}_B} = -1,$$

where the final equality uses the definition of $\overline{b}_B$; hence, for this region the optimality condition always holds.

By construction, for $b = \overline{b}_B$, condition (v) of Proposition B.1 is satisfied.

Note that as $V_B(\underline{b}_B) = \overline{V}$, the derivative of V_B is continuous at $\underline{b}_B$. The only point of non-differentiability is $b = 0$. In particular, note that $\lim_{b \downarrow 0} V_B'(b) = -\lim_{b \downarrow 0} q_B(b)(\overline{C} - \rho V_B(0))(\overline{C} - y)$. Hence, if $\lim_{b \downarrow 0} q_B(b) < 1$, then there is a convex kink at $b = 0$. This is consistent with condition (iii) in Proposition B.2.

Hence, the conditions of Proposition B.2 hold if and only if (22) holds. □

C.7 Proof of Proposition 4

Proof. There are three claims in the proposition:

Part (i). If a borrowing allocation is efficient, it must dominate the stationary allocation, hence

$$r P_B^{\star}(v) \geq y - \rho V$$

for any $V \geq \overline{V}$. From the expressions for $P_B^\star$ and V_B, this implies that:

$$
\begin{aligned}
V_B(b) &\geq \frac{y - rq_B(b)b}{\rho} \\
&= \frac{y - \frac{rq_B(b)}{r+\delta(1-q_B(b))}(r + \delta(1 - q_B(b)))b}{\rho} \\
&\geq \frac{y - (r + \delta(1 - q_B(b)))b}{\rho} \text{ for all } b \in [0, \underline{b}_B],
\end{aligned}
$$

where the last inequality follows from $rq_B(b) \leq r + \delta(1 - q_B(b))$ for all $\delta \geq 0$, $q_B(b) \leq 1$ and $b \geq 0$. And thus condition (22) is satisfied.

Part (ii). For $b \in [0, \underline{b}_B]$, Condition (22) becomes

$$\rho V_B(b) - (y - (r + \delta)b + \delta q_B(b)b) \geq 0.$$

Now, from the price equation (31), we have

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}}\right)^{\frac{r}{r+\delta}} = \frac{\overline{C} - y + rp}{\overline{C} - y + r\underline{p}}, \tag{43}$$

where $\underline{p} \equiv P_B^\star(\overline{V}) = \underline{q}\underline{b}_B$. From this expression, we can define $q_B(b) = F(\delta, p)$, holding the other parameters constant. Recall that condition (22) is restricted to $b \in [0, \underline{b}_B]$; hence, the domain of interest for p is $[0, \underline{p}]$, which is independent of δ. We shall use the fact that

$$\frac{\partial F(\delta, p)}{\partial \delta} = \frac{1 - F(\delta, p)}{r + \delta}\left(\underline{q} + \ln\left(\frac{1 - \underline{q}}{1 - F(\delta, p)}\right)\right), \tag{44}$$

keeping in mind that $\underline{q} = (r + \delta)/(r + \delta + \lambda)$ and hence varies with δ.

Let $V_B^\star$ denote the inverse of $P_B^\star$. Recall that $V_B(b) = V_B^\star(q_B(b)b)$. Condition (22) can be written:

$$G(\delta, p) \equiv \rho V_B^\star(p) - y + (r + \delta)p/F(\delta, p) - \delta p \geq 0.$$

Taking the derivative with respect to δ, we have that:

$$
\begin{aligned}
\frac{\partial G(\delta, p)}{\partial \delta} &= \frac{p}{F(\delta, p)} - p - \frac{(r + \delta)p}{F(\delta, p)^2}\frac{\partial F(\delta, p)}{\partial \delta} \\
&= \frac{p}{F(\delta, p)}\left(1 - F(\delta, p) - \frac{(r + \delta)}{F(\delta, p)}\frac{\partial F(\delta, p)}{\partial \delta}\right) \\
&= \frac{p(1 - F(\delta, p))}{F(\delta, p)^2}\left(F(\delta, p) - \frac{(r + \delta)}{(1 - F(\delta, p))}\frac{\partial F(\delta, p)}{\partial \delta}\right) \\
&= \frac{p(1 - F(\delta, p))}{F(\delta, p)^2}\left(F(\delta, p) - \underline{q} - \ln\left(\frac{1 - \underline{q}}{1 - F(\delta, p)}\right)\right).
\end{aligned}
$$

Note that $\partial G/\partial\delta \le 0$ if

$$F(\delta,p) - \underline{q} - \ln\left(\frac{1-\underline{q}}{1-F(\delta,p)}\right) \le 0.$$

For $p = \underline{p}$, $F(\delta,\underline{p}) = \underline{q}$, and this term is zero. Moreover, this expression is increasing in p as $\partial F/\partial p < 0$. Hence, $\partial G(\delta,p)/\partial\delta \le 0$ for $p \in [0,\underline{p}]$. Thus, if $G(\delta_0,p) \ge 0$, then $G(\delta,p) \ge 0$ for $\delta \in [0,\delta_0]$.

Part (iii). The fact that saving is efficient implies

$$\frac{y-\rho\overline{V}}{r} > P_B^\star(\overline{V}) = \underline{q}\underline{b}_B,$$

where the last equality follows from the definition of $\underline{b}_B$. By continuity, there exists a $V_0 > \overline{V}$ such that

$$\frac{y-\rho V_0}{r} > P_B^\star(V_0) \equiv p_0 < \underline{p},$$

where the last inequality follows from the fact that $P_B^\star$ is strictly decreasing. As $V_0 = V_B^\star(p_0)$ by definition of $V_B^\star$ as the inverse of $P_B^\star$, this is equivalent to

$$\rho V_B^\star(p_0) < y - rp_0.$$

Evaluated at $p = p_0$, condition (22) is

$$G(\delta,p_0) = \rho V_B^\star(p_0) - y + \frac{rp_0}{F(\delta,p_0)} + p_0\delta\left(\frac{1}{F(\delta,p_0)} - 1\right) \ge 0. \tag{45}$$

Note that $\lim_{\delta\to\infty}\underline{q} = 1$, and hence $F(\delta,p) \ge \underline{q}$ also converges to 1. Hence, $\rho V_B^\star(p_0) - y + rp_0/F(\delta,p_0) \to \rho V_B^\star(p_0) - y + rp_0 < 0$. We now show that the last term in (45) converges to zero; that is, $\delta(1-F(\delta,p_0)) \to 0$. From the definition of F in (43), we have

$$\delta(1-F(\delta,p_0)) = \frac{\lambda\delta}{r+\delta+\lambda}\left(\underbrace{\frac{\overline{C}-y+rp_0}{\overline{C}-y+r\underline{p}}}_{<1}\right)^{\frac{r+\delta}{r}}.$$

As the ratio raised to the power $(r+\delta)/r$ is strictly less than one as $p_0 < \underline{p}$, the right-hand side goes to zero as $\delta \to \infty$. Hence, there exists a δ_1 such that for all $\delta > \delta_1$, $G(\delta,p_0) < 0$, violating the condition for the borrowing equilibrium.

□

C.8 Proof of Proposition 5

Proof. We proceed to show the necessity and sufficiency parts of the proposition independently.

The "only if" part. Toward a contradiction, suppose that $V_S(\underline{b}_S) < V_B(\underline{b}_S)$ (or equivalently $\underline{b}_B > \underline{b}_S$), and the conjectured saving equilibrium is indeed an equilibrium. First, note that $q_S(b) \in [\underline{q},1]$, as the government defaults only with the arrival of $V^D = \overline{V}$.

By construction, $\hat{V}(\underline{b}_S) = V_S(\underline{b}_S) = \overline{V}$. As V_S is strictly decreasing, we have $V_S(\underline{b}_B) < \overline{V} = V_B(\underline{b}_B)$. Hence, V_S and V_B do not intersect in $[\underline{b}_S, \underline{b}_B]$, and $b^I > \underline{b}_S$, and $V_S(\underline{b}_B) = \hat{V}(\underline{b}_B)$.

We also have for $b \geq \underline{b}_B$: $\hat{V}'(b) = -q_S(b) \leq -\underline{q} \leq V_B'(b)$, where the latter inequality uses a property of the borrowing allocation value function (shown in the proof of Proposition 3). This implies that $\hat{V}(b) < V_B(b)$ for all $b \geq \underline{b}_B$, and there is no point of intersection to generate b^I and $V_S = \hat{V}$ for all $b \in [\underline{b}_S, \overline{b}_S]$. Now at $\overline{b}_S$, we must have (as an equilibrium requirement) that $\hat{V}(\overline{b}_S) = \underline{V} < V_B(\overline{b}_S)$, where the latter inequality follows from the fact that $\hat{V} < V_B$ on this domain. Thus, $\overline{b}_S < \overline{b}_B$. However, we have

$$
\begin{aligned}
(\rho + \lambda)\underline{V} &= y - [r + \delta(1 - \underline{q})]\overline{b}_B + \lambda\overline{V} \\
&< y - [r + \delta(1 - \underline{q})]\overline{b}_S + \lambda\overline{V} \\
&\leq y - [r + \delta(1 - q_S(\overline{b}_S)]\overline{b}_S + \lambda\overline{V} \\
&= (\rho + \lambda)\hat{V}(\overline{b}_S) = (\rho + \lambda)\underline{V},
\end{aligned}
$$

where the first line uses the definition of $\overline{b}_B$; the second line uses $\overline{b}_B > \overline{b}_S$; the third uses $q_S(b) \geq \underline{q}$; and the final two equalities use the fact that $\hat{V}(b)$ is the stationary value at price q_S and the definition of $\overline{b}_S$, respectively. Hence, we have generated a contradiction.

The "if" part. We first verify that V_S satisfies the conditions of Proposition B.2 and establish that q_S is a valid equilibrium price schedule.

First, consider the government's problem.

Condition (iv) and (v) of Proposition B.2 are satisfied by construction. For $b = \underline{b}_S$, condition (ii) of Proposition B.2 applies and is satisfied by construction.

For $b < \underline{b}_S$, the conjectured value function is differentiable. For the HJB to hold with $c = \overline{C}$ given that $q_S(b) = 1$ in this region, we require $V_S'(b) \geq -q_S(b) = -1$. On this domain, $V_S(b) = V_S^\star(b)$, where the latter is the inverse of the efficient solution $P_S^\star$. As $P_S^{\star\prime}(v) \leq -1$, we have $V_S'(b) \geq -1 = -q_S(b)$. Hence, $c = \overline{C}$ is indeed optimal, and the HJB holds with equality.

For $b \in (\underline{b}_S, b^I)$, $V_S(b) = \hat{V}(b)$, and thus V_S is differentiable and satisfies the HJB with equality by construction. Note that if $q_S(b) \geq \overline{q}$ (something we check below), then $\dot{b} \leq 0$ in (b_S, b^I) by equation (16) (consistent with the equilibrium conjecture that the government is saving in this region). This implies that $C_S(b) = \hat{C}(b) \leq \overline{C}$, and thus the conjectured policy function is a valid one (recall that we are assuming that $\underline{C}$ is sufficiently low and can thus be ignored as a constraint).

For $b \in (b^I, \overline{b}_S)$, $V_S(b) = V_B(b)$ and differentiability of V_B implies that V_S is differentiable. The proof of Proposition 3 establishes that the HJB holds with equality in this domain, given that $q^S(b) = \underline{q}$.

This confirms that condition (i) of Proposition B.2 holds.

If $b^I \in (\underline{b}_S, \overline{b}_S)$, $V_S(b^I) = V_B(b^I)$, and there is a potential point of non-differentiability at b^I. If $q_S(b^I) \geq \underline{q}$ (something that we check below), we have that this kink is convex. Thus, condition (iii) of Proposition B.2 holds.

Hence, given the conjectured q_S, the value function is a viscosity solution to the government's HJB equation.

Next, let us consider the price function. The only thing left to check is that $q^S(b) \in [\underline{q}, 1]$ for $b \in (\underline{b}_S, b^I)$, where $b^I \in (\underline{b}_B, \overline{b}_B)$. In this region, $q^S(b) \leq 1$ by equation (35). In addition,

$$\begin{aligned}(\rho+\lambda)V_S(b) &= y - \left[r + \delta(1-q_S(b))\right]b + \lambda\overline{V} \\ &\geq (\rho+\lambda)V_B(b) \\ &\geq y - \left[r + \delta(1-\underline{q})\right]b + \lambda\overline{V},\end{aligned}$$

where the first equality and second inequality follow from the equilibrium construction on $b \in (\underline{b}_S, b^I)$. The last inequality follows from the construction of $V_B(b)$ for $b \in (\underline{b}_B, \overline{b}_B)$. Comparison of the first and last lines establishes that $q_S(b) \geq \underline{q}$. □

C.9 Proof of Proposition 6

Proof. The fact that the efficiency of saving is a necessary condition for a saving equilibrium is established in the text. Turning to equation (25), multiply both sides of equation (24) by $\underline{q}$ to obtain the following necessary and sufficient condition:

$$\underline{q}P_S^\star(\overline{V}) \geq \underline{q}\underline{b}_B = P_B^\star(\overline{V}).$$

Using $\underline{q} = (r+\delta)/(r+\delta+\lambda)$ and the fact that $P_S^\star(\overline{V}) > P_B^\star(\overline{V})$, we solve for δ to obtain

$$\delta \geq \frac{\lambda P_B^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} - r = \frac{(r+\lambda)P_B^\star(\overline{V}) - rP_S^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} \geq 0,$$

where the last inequality is strict when $\rho > r$, as seen in the definition of $P_B^\star$. Thus, this is a necessary and sufficient condition for the saving equilibrium, proving the proposition. □

C.10 Proof of Proposition 7

Proof. Note that $P_B^\star(v)$ is increasing in $\overline{C}$. Hence,

$$\begin{aligned}P_B^\star(\overline{V}) &\leq \lim_{\overline{C}\to\infty} P_B^\star(\overline{V}) \\ &= \frac{y - \rho\overline{V} + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda} \\ &= \frac{rP_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda}.\end{aligned}$$

Then a sufficient condition for saving to be strictly efficient is

$$P_S^\star(\overline{V}) > \frac{rP_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda},$$

or

$$\frac{\lambda}{\rho - r} > \frac{r(\overline{V} - \underline{V})}{y - \rho\overline{V}},$$

which is the last inequality in the proposition.

Similarly,

$$P_S^\star(\overline{V}) - P_B^\star(\overline{V}) \geq \frac{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})}{r + \lambda},$$

and

$$\begin{aligned}\frac{\lambda P_B^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} - r &\leq \frac{\lambda \left(r P_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V}) \right)}{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})} - r \\ &= \frac{(r + \lambda)(\rho - r)(\overline{V} - \underline{V})}{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})} \equiv \underline{\delta}.\end{aligned}$$

From Proposition 6, a sufficient condition for a saving equilibrium, given that saving is efficient, is that δ is greater than $\underline{\delta}$. Note that $\underline{\delta}$ is strictly positive and independent of $\overline{C}$.

For the borrowing equilibrium, we need to show that the condition in equation (22) is satisfied as $\overline{C}$ becomes arbitrarily large. Specifically, fix any $\delta = \overline{\delta} > \underline{\delta}$. Define

$$A(b) \equiv \rho V_B(b) - (y - [r + \delta(1 - q_B(b))]b),$$

where A implicitly depends on $\overline{C}$ and δ. Note by condition (22) in Proposition 3 that if $A(b) > 0$ on $[0, \overline{b}_B] \supseteq [0, \underline{b}_B]$, then a borrowing equilibrium exists.

To establish the properties of $A(b)$ as $\overline{C} \to \infty$, first note that $\overline{b}_B$ is independent of $\overline{C}$. In addition,

$$\lim_{\overline{C} \to \infty} V_B(b) = \underline{V} + \underline{q}(\overline{b}_B - b),$$

where we use the fact that $q_B(b) \to \underline{q}$ for $b \in [0, \overline{b}_B]$ as $\overline{C} \to \infty$. As the point-wise limit is continuous in b, and by Lemma B.2 $V_B(b)$ is monotonic given $\overline{C}$, the convergence is uniform on the compact set $[0, \overline{b}_B]$ (see Theorem A of Buchanan and Hildebrandt (1908)).

Similarly, for $b \in [0, \overline{b}_B]$,

$$\lim_{\overline{C} \to \infty} \{y - [r + \delta(1 - q_B(b))]b\} = y - [r + \delta(1 - \underline{q})]b = y - (r + \lambda)\underline{q}b.$$

Again, the convergence is uniform by the same logic.

Hence, $A(b)$ converges uniformly on $[0, \overline{b}_B]$ to

$$\overline{A}(b) \equiv \lim_{\overline{C} \to \infty} A(b) = \rho \underline{V} + \rho \underline{q}(\overline{b}_B - b) - (y - (r + \lambda)\underline{q}b).$$

We now establish that $\overline{A}(b) > 0$ for $b \in [0, \overline{b}_B]$. The linearity of $\overline{A}(b)$ in b implies that if the inequality holds for $b = 0$ and $b = \overline{b}_B$, it is satisfied for all intermediate points. For $b = 0$, we have

$$\begin{aligned}\overline{A}(0) &= \rho \underline{V} + \rho \underline{q} \overline{b}_B - y \\ &= \frac{(\rho - r - \lambda)(y - \rho \underline{V}) + \rho\lambda(\overline{V} - \underline{V})}{r + \lambda} \\ &= \frac{(\rho - r - \lambda)(y - \rho \overline{V}) + (\rho - r)\rho(\overline{V} - \underline{V})}{r + \lambda} > 0,\end{aligned}$$

where the second line uses the definition of $\underline{q}\overline{b}_B$ and the final inequality uses the condition in the proposition. Similarly,

$$\begin{aligned}\overline{A}(\overline{b}_B) &= \rho\underline{V} - (y - (r+\lambda)\underline{q}\overline{b}) \\ &= \lambda(\overline{V} - \underline{V}) > 0.\end{aligned}$$

Hence, $\min_{b\in[0,\overline{b}_B]} \overline{A}(b) = \min\{\overline{A}(0), \overline{A}(\overline{b})\} > 0$.

As $A \rightarrow \overline{A}$ uniformly on $[0, \overline{b}_B]$, for every $\epsilon > 0$, there exists an M such that for all $\overline{C} > M$, we have $A(b) > \overline{A}(b) - \epsilon$ for $b \in [0, \overline{b}_B]$. Setting $\epsilon < \min_{b\in[0,\overline{b}_B]} \overline{A}(b)$, we have $A(b) > 0$ for all $b \in [0, \overline{b}_B]$ and $\overline{C} > M$. By Proposition 3, this is sufficient to establish the existence of a borrowing equilibrium for $\delta = \overline{\delta}$ when $\overline{C} > M$. By part (ii) of Proposition 4, we have a borrowing equilibrium for all $\delta \in [0, \overline{\delta}]$.

Combining results, there exists a non-empty interval $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ and M such that for all $\overline{C} > M$ and $\delta \in \Delta$, both saving and borrowing equilibria coexist. □

C.11 Proof of Proposition 8

Proof. We first sketch out the borrowing equilibrium under the assumed policy. Let $\{V_B^P, q_B^P\}$ denote the equilibrium policy and price functions. The conjectured policy is for the government to borrow to $\overline{b}_B$, which is the endogenous limit in the borrowing equilibrium absent the policy. From (27), it is optimal for the bondholders to sell their bonds at price $q^* > \underline{q}$ as soon as $b = \underline{b}_B^P$, where the latter is defined by $V^P(\underline{b}_B^P) = \overline{V}$. That is, bondholders sell their bonds to the third party as soon as debt enters the Crisis Zone. We have

$$\begin{aligned}V_B^P(\overline{b}_B) &= \frac{y - [r + \delta(1-q^*)]\overline{b}_B + \lambda\overline{V}}{\rho+\lambda} \\ &= \underline{V} + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho+\lambda}.\end{aligned}$$

The last term reflects that the government rolls over debt at q^* rather than $\underline{q}$ once it reaches the borrowing limit. The expression assumes that the government defaults upon the arrival of $\overline{V}$. To see that this is optimal, note that the alternative of never defaulting yields the value

$$\frac{y - [r + \delta(1-q^*)]\overline{b}_B}{\rho} \leq \frac{y - r\overline{b}_B}{\rho} < \frac{y - r\underline{b}_S}{\rho} = \overline{V}.$$

Facing $q_B^P(b) = q^*$ in the Crisis Zone, the government's value can be obtained from the HJB, and it is straightforward to verify that the first-order condition for $c = \overline{C}$ holds on this domain. As $q^* > \underline{q}$, $\underline{b}_B^P > \underline{b}_B$, where the latter is the benchmark borrowing equilibrium's threshold for the Safe Zone. Note as well that $q^* > \underline{q}$ implies that the third party takes a loss in expectation in the Crisis Zone.

For $b \in [0, \underline{b}_P]$, bondholders purchase debt at price $q_B^P(b)$ and collect r plus maturing principal until $b = \underline{b}_B^P$, at which point they sell at q^*. The equilibrium is recovered by solving the system of differential equations:

$$\rho V_B^P(b) = \overline{C} + V_B^{P\prime}(b)\dot{b}$$
$$(r+\delta)q_B^P(b) = r + \delta + q_B^{P\prime}(b)\dot{b}$$
$$\dot{b} = \frac{\overline{C} + (r+\delta)b - y}{q_B^P(b)} - \delta b,$$

with the boundary conditions $V_B^P(\underline{b}_B^P) = \overline{V}$ and $q_B^P(\underline{b}_B^P) = q^*$. Note that these equations are identical to those in the benchmark borrowing equilibrium except that the boundary condition $\underline{b}_B^P > \underline{b}_B$ and $q^* > \underline{q}$.

As is the case in the benchmark equilibrium, a necessary and sufficient condition for V_B^P to be a solution to the government's problem when facing q_B^P is

$$V_B^P(b) \geq \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho},$$

for all $b \in [0, \underline{b}_B]$. Following the same approach as in the proof of Proposition 7, we show that this inequality holds as $\overline{C} \to \infty$ uniformly over the full debt domain $[0, \overline{b}_B]$.

As $\overline{C} \to \infty$, we have for $b \in [0, \overline{b}_B]$,

$$\lim_{\overline{C}\to\infty} V_B^P(b) = V_B^P(\overline{b}_B) + q^*(\overline{b}_B - b)$$
$$\lim_{\overline{C}\to\infty} \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} = \frac{y - [r + \delta(1 - q^*)]b}{\rho}.$$

Recall from the proof of Proposition 7, that

$$\overline{A}(b) = \underline{V} + \underline{q}(\overline{b}_B - b) - \frac{y - [r + \delta(1 - \underline{q})]b}{\rho} \geq 0$$

under the conditions of the proposition. Note that this implies

$$\lim_{\overline{C}\to\infty} \left(V_B^P(b) - \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \right) = \overline{A}(b) + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} + (q^* - \underline{q})(\overline{b}_B - b) - \frac{\delta(q^* - \underline{q})}{\rho} b.$$

This expression is linear in b, and hence it is sufficient to verify the inequality at the endpoints $b = 0$ and $b = \overline{b}_B$. The fact that $\overline{A}(0) > 0$ and $q^* > \underline{q}$ implies that the limit is strictly positive at $b = 0$. For $b = \overline{b}_B$, we have

$$\begin{aligned}
&\overline{A}(\overline{b}_B) - \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\overline{b}_B \\
&= \frac{y - [r + \delta(1 - \underline{q})]\overline{b}_B + \lambda\overline{V}}{\rho + \lambda} - \frac{y - [r + \delta(1 - \underline{q})]\overline{b}_B}{\rho} + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\overline{b}_B \\
&= \frac{-\lambda}{\rho(\rho + \lambda)} \left(y - [r + \delta(1 - q^*)]\overline{b}_B - \rho\overline{V} \right) = \frac{-\lambda}{\rho(\rho + \lambda)} \left(r(\underline{b}_S - \overline{b}_B) - \delta(1 - q^*)\overline{b}_B \right) > 0,
\end{aligned}$$

where the last inequality uses $\overline{b}_B > \underline{b}_S$. This completes the proof of part (i).

For part (ii), the saving equilibrium requires $V_B^P(\underline{b}_S) \leq \overline{V}$. As $\overline{C} \to \infty$,

$$\begin{aligned} V_B^P(\underline{b}_S) &= V_B^P(\overline{b}_B) + q^*(\overline{b}_B - \underline{b}_S) \\ &= \frac{y - r\overline{b}_B + \lambda\overline{V}}{\rho + \lambda} + \overline{b}_B - \underline{b}_S - (1 - q^*)(\overline{b}_B - \underline{b}_S) - \frac{\delta(1 - q^*)\overline{b}_B}{\rho + \lambda} \\ &= \overline{V} + \frac{(\rho + \lambda - r)(\overline{b}_B - \underline{b}_S)}{\rho + \lambda} - (1 - q^*)\left(\overline{b}_B - \underline{b}_S + \frac{\delta\overline{b}_B}{\rho + \lambda}\right). \end{aligned}$$

As the second term is strictly positive, there exists a $\tilde{q} < 1$ such that this expression exceeds $\overline{V}$ for $q^* > \tilde{q}$, hence violating the necessary condition for a saving equilibrium.

□

C.12 Proof of Proposition 9

Proof. For part (i), note that in the saving equilibrium undistorted by policy, $q_S(b) = 1$ for $b \leq \underline{b}_S$. Hence, imposing a price floor restricted to the Safe Zone does not alter the saving equilibrium, which exists by Proposition 7. Hence, the price floor is irrelevant under the saving equilibrium.

Using the notation introduced in the proof of Proposition 8, a necessary condition for the borrowing equilibrium under the policy is for $b \in [0, \underline{b}_B^P]$

$$V_B^P(b) \geq \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \geq \frac{y - [r + \delta(1 - q^*)]b}{\rho}.$$

Recall that in the construction of the borrowing equilibrium, $\underline{b}_B$ is defined by solving the HJB assuming $q_B(b) = \underline{q}$. Hence, $V_B^P(b) = V_B(b)$ for $b > \underline{b}_S$, as the policy is restricted to $b \in [0, \underline{b}_S]$. As $V_B(\underline{b}_S) < \overline{V}$ by the inequality of Proposition 7, we have $\underline{b}_B^P < \underline{b}_S$. For $b = \underline{b}_B^P$, we have

$$V_B^P(\underline{b}_B^P) = \overline{V} = \frac{y - r\underline{b}_S}{\rho} < \frac{y - r\underline{b}_B^P}{\rho},$$

where the first two equalities use the definitions of $\underline{b}_B^P$ and $\underline{b}_S$, respectively. Hence, there exists a $\hat{q} < 1$ such that

$$V_B^P(\underline{b}_B^P) < \frac{y - [r + \delta(1 - q^*)]\underline{b}_B^P}{\rho},$$

for $q^* > \hat{q}$, violating the necessary condition for a borrowing equilibrium. This proves part (ii).

□

Appendix D The Hybrid Equilibrium

In this appendix, we present a third type of competitive equilibrium, which we label the "hybrid" equilibrium because it combines features of both borrowing and saving equilibria. In particular, the government never saves, as in the borrowing equilibrium, but part of the Safe Zone is absorbing, as in the saving equilibrium. The main purpose of introducing the hybrid equilibrium is to show existence of a competitive equilibrium; in particular, we prove that if neither the borrowing nor the saving equilibrium exists, then a hybrid equilibrium exists. The equilibrium objects are depicted in Figure D.1 using the same parameters as in Figures 1 and 3.

More formally, given V_B in (32), define the threshold

$$V_B(b_H) = \frac{y - rb_H}{\rho}, \tag{46}$$

if such a threshold exists on the domain $[0, \underline{b}_B] \cap [0, \underline{b}_S]$. The equilibrium conjecture is that for $b \leq b_H$, the government borrows up to b_H and then remains there indefinitely. This behavior is similar to the Safe Zone policy in the saving equilibrium, but the threshold b_H may be strictly below $\underline{b}_S$. At b_H, given that $V_B(\underline{b}_H) = (y - rb_H)/\rho$, the government is indifferent to remaining at b_H at risk-free prices versus borrowing to the debt limit at the borrowing equilibrium price schedule. The conjecture is that for $b > b_H$, the government borrows. In a hybrid equilibrium, therefore, b_H is a stationary point that is stable from the left but not the right.

For $b < b_H$, we solve the government's HJB assuming $c = \overline{C}$ to obtain a candidate V_H on this domain, using the boundary condition $\rho V_H(\underline{b}_H) y - rb_H$. For $b > b_H$, the hybrid equilibrium coincides with the borrowing equilibrium. Setting $\overline{b}_H \equiv \overline{b}_B$, the hybrid equilibrium value function is therefore

$$V_H(b) = \begin{cases} \dfrac{\overline{C} - (\overline{C} + rb_H - y)\left(\frac{\overline{C} + rb - y}{\overline{C} + rb_H - y}\right)^{\frac{\rho}{r}}}{\rho} & \text{for } b \leq b_H \\ V_B(b) & \text{for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{47}$$

The associated price schedule is

$$q_H(b) = \begin{cases} 1 \text{ for } b \leq b_H \\ q_B(b) \text{ for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{48}$$

Finally, the policy function for consumption is

$$C_H(b) = \begin{cases} \overline{C} \text{ for } b < b_H \\ y - rb_H \text{ for } b = b_H \\ C_B \text{ for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{49}$$

We state the following:

Proposition D.3. *Suppose neither the borrowing equilibrium nor the saving equilibrium exists. Specifically, suppose that $\underline{b}_S < \underline{b}_B$ and that there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b}$. Then a hybrid equilibrium exists.*

Proof. The conjectured price schedule q_H is consistent with the lenders' break-even condition given the assumed government policy. Thus, to establish the conditions of an equilibrium, it is sufficient to prove that V_H is a solution to the government's HJB.

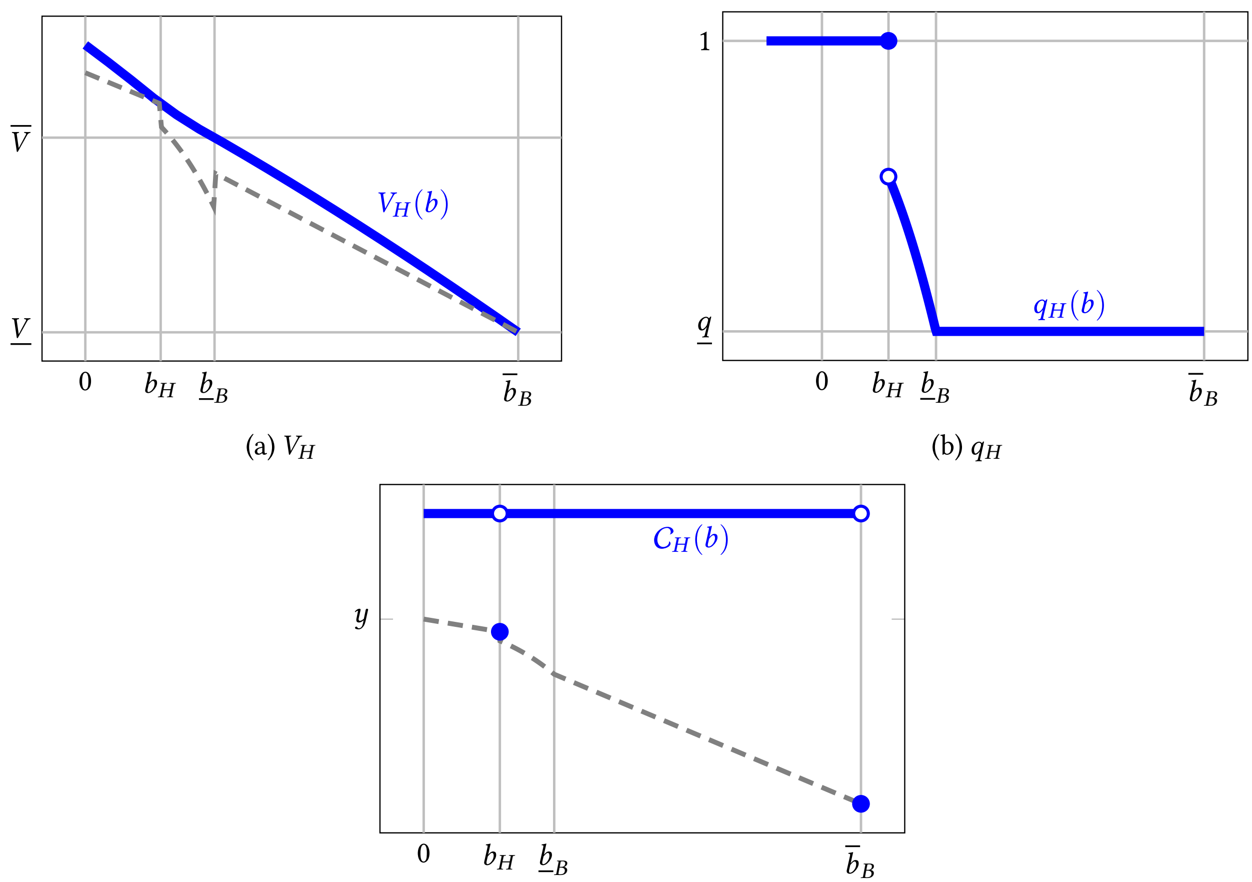
(i) For $b \in [\underline{b}_S, \overline{b}_B]$: By premise, $\underline{b}_B > \underline{b}_S$. This implies that $\rho V_B(\underline{b}_S) > \overline{V} = y - r\underline{b}_S \geq y - [r + \delta(1 - q_B(\underline{b}_S))]\underline{b}_S$. For $b > \underline{b}_S$, we have $\rho\overline{V} > y - rb$. As $V_B \geq \overline{V}$ for $b \leq \underline{b}_B$, we have $V_B(b) \geq y - rb$ for $b \in [\underline{b}_S, \underline{b}_B]$. From the proof of Proposition 3, this implies that $V_H(b) = V_B(b)$ satisfies the government's HJB on this domain. The proof of Proposition 3 extends this to $b \in [\underline{b}_B, \overline{b}_B]$ as well.

(ii) For $b \leq \underline{b}_S$: Note that the premise implies there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b} \leq y - r\hat{b}$. The above established that $\rho V_B(b) > y - rb$ for $b \in [\underline{b}_S, \overline{b}_B]$. Hence, $\hat{b} < \underline{b}_S$. By continuity, there exists a $b_H \in (\hat{b}, \underline{b}_S)$ such that $\rho V_B(b_H) = y - rb_H$. Note as well that this implies $V'_B(b_H) = \lim_{b \downarrow b_H} V'_H(b) \geq -r/\rho$. From the expression for V_H, we have $\lim_{b \uparrow b_H} V_H(b_H) = -1 \leq -r/\rho$. Hence, V_H is either differentiable or has a convex kink at b_H, satisfying the conditions for a solution to the government's HJB at b_H. For $b < b_H$, $V'_H(b) \geq -1$, implying that the HJB is satisfied on this domain as well. Finally, $V_H(b) > \overline{V}$ for $b \leq b_H$, rationalizing the government's non-default on this domain.

□

This establishes that at least one of the three types of equilibria always exists. We note that the hybrid may coexist with the other equilibria as well. In fact, as $\overline{C} \to \infty$, the condition for multiplicity presented in Proposition 7 also implies the existence of a hybrid equilibrium.

Figure D.1: Hybrid Equilibrium



(a) V_H

(b) q_H

(c) C_H

Appendix E The Quantitative Model

In this appendix, we provide additional details on the quantitative exercise discussed in Section 7. As noted, we follow Chatterjee and Eyigungor (2012) closely, and therefore provide only the main components.

Time is discrete and the model is calibrated to quarterly frequency.

Output process. A small open economy receives an endowment that is comprised of a persistent process y_t plus an *iid* shock m_t. As in CE12, m is drawn from a truncated Normal with support $[-\overline{m}, \overline{m}]$, mean zero, and variance σ_m^2. We follow CE12 and set $\overline{m} = 0.006$ and $\sigma_m = 0.003$. For the persistent component, begin with the AR(1) process:

$$\ln y_t = (1-\rho)\mu + \rho \ln y_{t-1} + \varepsilon_t,$$

where ε has a mean-zero Normal distribution with variance σ_ε^2. We approximate this process with a 200-element grid spanning 6 standard deviations following the standard methodology of Tauchen.

We augment this process by adding a "disaster state" y_{dis}. The transition probabilities to and from the disaster state follow: $\Pr[y_{t+1} = y_{dis} | y_t \neq y_{dis}] = \pi_{dis}$ and $\Pr[y_{t+1} \neq y_{dis} | y_t = y_{dis}] = \pi_{rec}$. Conditional on transitioning from disaster to non-disaster regimes, the non-disaster endowment is drawn from the non-disaster grid assuming probabilities computed from a discretized Normal distribution with mean $(1-\rho)\mu + \rho y_{dis}$ and variance σ_ε^2. While in the disaster state, we set $m = 0$.

Following Barro and Ursa (2008), we set π_{dis} to be 0.97%, and y_{dis} to be 0.20 log points below the conditional mean of the normal AR(1) process. Barro and Ursa (2008) estimates the average length of a disaster to be 3.5 years, and hence we set $\pi_{rec} = 7.14\%$ in our quarterly model. μ is set to normalize the unconditional mean of $ln(y_t)$ to 0. We then select ρ and σ_ε to ensure that the combined AR(1)-plus-disaster-shock endowment process generates an autocorrelation coefficient of 0.949 and an innovation variance of 0.027, which are the targets used by CE12 to replicate their Argentina GDP sample. The resulting values are $\rho = 0.931$ and $\sigma_\varepsilon = 0.0178$. The unconditional mean endowment is 1.03.

Default cost. The persistent endowment process for a government in default status, y^d, is given by:

$$y_t^d = \begin{cases} y_t - \max\left\{0, d_0 y_t + d_1 y_t^2\right\} & \text{if } y_t \neq y_{dis} \\ (1-d_{dis}) y_{dis} & \text{otherwise.} \end{cases}$$

The first-line on the right-hand side concerns non-disaster states and is the functional form used by CE12; we follow them by setting $d_0 = -0.188$ and $d_1 = 0.246$. The second line concerns the disaster state, and, as discussed in Section 7, $d_{dis} = 0.045$. Following CE12 we assume the government exits default status with constant hazard rate 0.0385. Following CE12, while in default, if $y_t \neq y_{dis}$, then m_t is drawn from its usual process – the one exception being in the first period of default, in which case $m_t = -\overline{m}$ for computational reasons.

Government objective. The government has a discount factor $\beta = 0.95402$ and utility $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma = 2$. Again, we follow CE12 for the values of β and γ.

Bonds, coupons and maturity. Bonds mature with probability δ and pay a coupon κ every period prior to maturity. In the benchmark, we follow CE12 and set $\delta = 1/20$ and $\kappa = 0.03$. Lenders are risk-neutral and are willing to borrow and lend at an expected net interest rate of $r = 0.01$. The risk-free price is therefore:

$$q^\star = \frac{\delta + (1-\delta)\kappa}{r + \delta} = 1.308.$$

Whenever we recompute the model with different maturity, we adjust κ to keep $q^\star$ at this value.

Computation Algorithm. To compute the model, we let debt choices be on a discrete grid contained on the interval $[0, 1.89]$.[34]

The computational algorithm for a saving equilibrium is motivated by our theoretical analysis. It begins by conjecturing that there exists a Safe Zone, where the price is risk free; that is, a $\underline{b}_S$ such that the equilibrium price $q(y, b) = q^\star$ for all $b \le \underline{b}_S$. For a candidate $\underline{b}_S$, we compute the value of repayment for $b \le \underline{b}_S$ imposing that debt remains below $\underline{b}_S$ and the price is the risk-free price.

This restricted government problem is solved using value function iteration. Let $V_S(b, y)$ denote the associated value function. In a saving equilibrium, $\underline{b}_S$ is the debt level that equates the value of repayment to the value of default in the disaster state. Hence, let $V^D(y_{dis}) \equiv V_S(\underline{b}_S, y_{dis})$, and associated with this value of default is a punishment d_{dis} in terms of lost endowment.

Given such value of $\underline{b}_S$, we solve the government's problem on the rest of the domain $b \in (\underline{b}_S, \overline{b}]$, this time without restricting the choice set for b, and taking as given the previously obtained values and (risk-free) prices for $b \le \underline{b}_S$. We follow the standard procedure of iterating on both values and prices until convergence. The final step is to check whether, given the equilibrium price schedule q computed for the entire domain, the government prefers to remain in the Safe Zone. This is done by recomputing the government's problem given the conjectured equilibrium q and verifying the value and price functions are consistent with optimization and the lenders' break-even constraint. This completes the construction for a candidate $\underline{b}_S$.

To compute the borrowing equilibrium, we follow the same algorithm of CE12.[35]

Results. Figure E.1 and E.2 show the debt policy and price functions for the computed equilibria. As can be seen in Figure E.1, the Safe Zone in the saving equilibrium is larger than in the borrowing. The policy functions in the saving equilibrium are such that if we start in the Safe Zone, debt never exits the Safe Zone. The policy functions in the borrowing equilibrium are such that if we start in the Safe Zone, debt will eventually leave it. Hence, just as in the theoretical analysis and shown in Figure E.2, the Safe Zone in the saving equilibrium features a risk-free price, while the Safe Zone in the borrowing equilibrium features a price strictly below the risk-free price, reflecting the future risk of default.

Table E.1 reports the key business cycle moments for our two equilibria. To obtain these, we simulate the model 1,000 times for 20,000 periods. We discard the first 1,000 periods of each sample. If the government defaults in period t and regains access in period $t + k$, we discard simulated observations for $[t, t + k + 20]$. This is the same procedure used by CE12.

Repeating this numerical exercise for different values of the maturity parameter, we confirm that the two equilibria exist for maturity values between 9 and 33 quarters (that is, for all $\delta \in \{1/33, 1/32, .., 1/9\}$).

The moments from the saving equilibrium reflect that the Safe Zone is absorbing. Recall that the Safe Zone has an upper threshold of 1.16, which is the same as the unconditional mean. This reflects the relative impatience of the government with respect to the risk free interest rate.

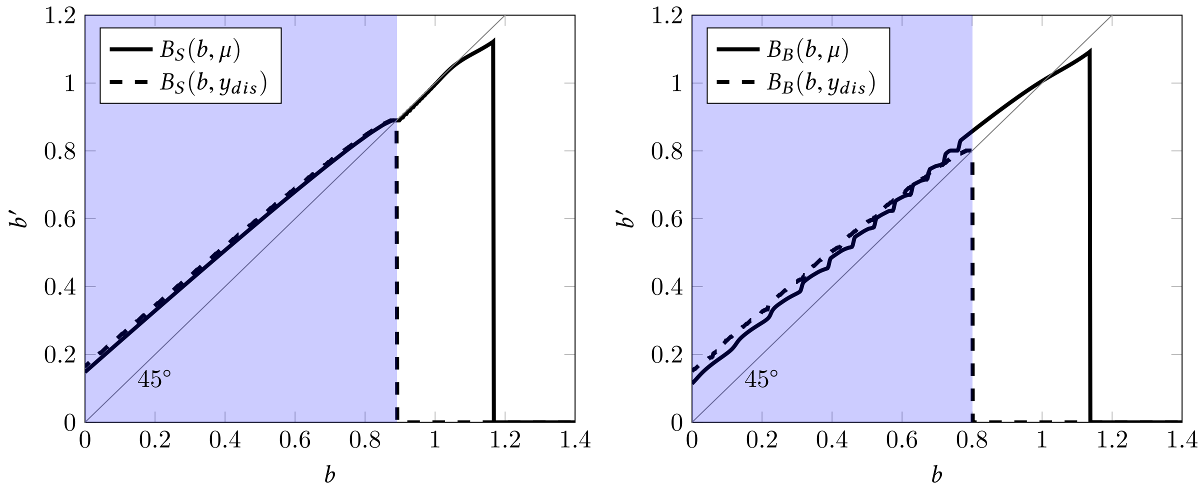
The moments from the borrowing equilibrium resemble those typically seen in the literature, such as those reported in CE12. As noted in the text, the presence of a disaster state and the associated punishment for default in that state support a higher mean debt-to-income ratio than CE12. One notable difference is that spreads are positively correlated with the endowment, while in CE12 they are negatively correlated (as they are in the data). This reflects that in our environment, this correlation is driven by the difference in spreads between the normal endowment regime and the disaster state.[36]

[34] For the computation, we use 946 points for this grid, including the boundaries.

[35] Specifically, we initialize prices at zero and use a very high smoothing (> 0.95) parameter when updating.

[36] In the disaster state, the government defaults if the face value of debt is greater than 0.80, when the mean face value in the simulation is 0.96. Thus the disaster-state spread is either close to zero (when debt is low) or undefined (when debt is high), generating a positive correlation between the spread and the endowment overall. When we

Figure E.1: Simulation Results: Policy Functions

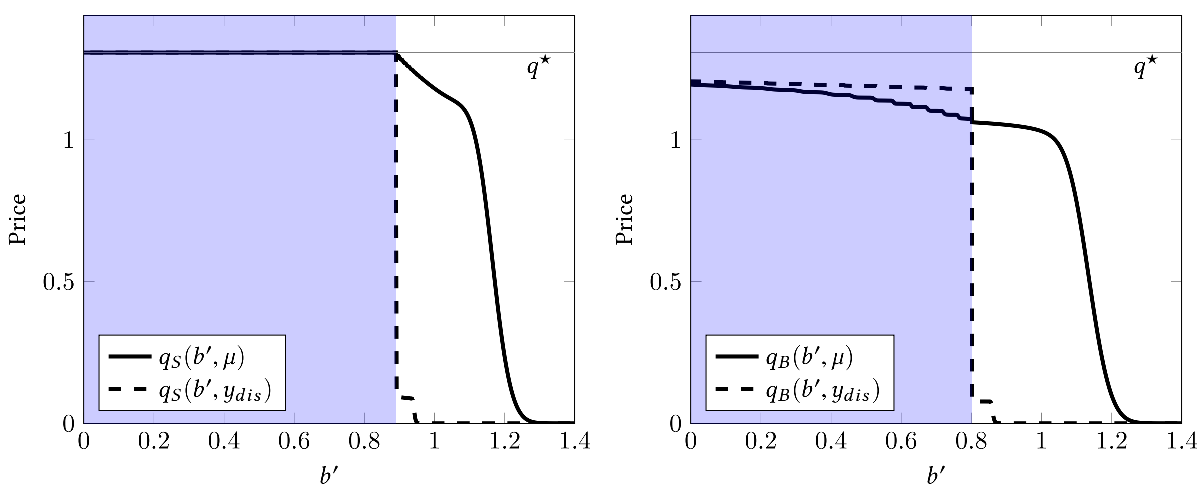


(a) Saving Equilibrium

(b) Borrowing Equilibrium

The figure depicts the debt policy functions for the computed saving and borrowing equilibrium. Each panel shows the equilibrium borrowing policy functions as a function of the current debt level, b, for two values of the endowment state: the mean of the AR1 process (solid), μ, and the disaster state, y_{dis} (dashed). The shaded area represents the Safe Zone – the region of debt such that default does not occur this period for any realization of the endowment. Each of the policy functions in both panels are averaged across potential realizations of the m shock conditional on no-default. The vertical portion of each policy function represents the point after which default is optimal in that state for all realizations of the m shock.

Figure E.2: Simulation Results: Price Functions



(a) Saving Equilibrium

(b) Borrowing Equilibrium

The figure depicts the price functions for the computed saving and borrowing equilibrium. Each panel shows the price functions as a function of the next period debt level, b', for two values of the endowment state: the mean of the AR1 process (solid), μ, and the disaster state, y_{dis} (dashed). The shaded area represents the Safe Zone – the region of debt such that default does not occur next period for any realization of the endowment.

condition on non-disaster states, the correlation between spreads and the endowment changes sign and is −0.2,

Table E.1: Business Cycle Moments for Quantitative Model

Moment	Saving Equilibrium	Borrowing Equilibrium
$E(r - r^f)$	0	0.070
$\sigma(r - r^f)$	0	0.010
$\sigma(\log(c))/\sigma(\log(y))$	1.01	1.07
$\sigma(nx/y)$	0.001	0.014
$\text{corr}(nx/y, \log(y))$	-0.99	-0.15
$\text{corr}(r - r^f, \log(y))$	NA	0.23
$\text{corr}(r - r^f, nx/y)$	NA	0.69
E(Face Value Debt/GDP)	0.88	0.96
E(Market Value Debt/GDP)	1.16	0.99
Default Frequency (Quarterly)	0	0.016

Note: The values of y and c refer to the levels of output (inclusive of m) and consumption. Net exports (nx) is defined as $y - c$. All relevant moments use the annualized spreads, denoted by $r - r^f$. The operators E, σ and $corr$ refer to the mean, standard deviation, and correlation computed following the methodology of Chatterjee and Eyigungor (2012).

closer to values discussed in CE12.