Online Appendix to "Self-Fulfilling Debt Dilution"

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1 Viscosity Solutions on Stratified Domains and the Proofs of Propositions B.1 and B.2

In this appendix, we establish the equivalence between the sequence problems and the viscosity solutions of the Hamilton-Jacobi-Bellman (HJB) equations. The two complications are that the objective and/or the dynamics are not necessarily continuous in the state variables. We rely on the results of Bressan and Hong (2007) (henceforth, BH) to establish the validity of the recursive formulation. This appendix introduces their environment and summarizes their core results. Relative to BH, we make minor changes in notation and consider a maximization problem while the original BH studies minimization. We then prove Propositions B.1 and B.2.

1.1 The Environment of Bressan and Hong (2007)

Let $X \subset \mathbb{R}^N$ denote the state space. In the benchmark BH environment, $X = \mathbb{R}^N$; however, they show how to restrict attention to an arbitrary subset by extending the dynamics and payoff functions to $\mathbb{R}^N$ such that the subset is an absorbing region. Let $\alpha(t) \in \mathcal{A}$ denote the control function, where $\mathcal{A}$ is the set of admissible controls. Dynamics of the state vector x are given by $\dot{x}(t) = f(x(t), \alpha(t))$.

Given a discount factor β and a flow payoff $\ell(x, \alpha)$, the sequence problem is

$$W(\bar{x}) = \sup_{\alpha \in \mathcal{A}} \int_0^\infty \ell(x(t), \alpha(t)) dt \tag{1}$$

$$\text{subject to } \begin{cases} x(0) & = \bar{x} \in X \\ \dot{x}(t) & = f(x(t), \alpha(t)). \end{cases}$$

The complication BH address is that f may not be continuous in x. In particular, assume there is a decomposition $X = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_M$ with the following properties. If $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_i = \emptyset$. In addition, if $\mathcal{M}_j \cap \overline{\mathcal{M}}_k \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}}_k$, where $\overline{\mathcal{M}}_k$ is the closure of $\mathcal{M}_k$.

BH's assumption $\boldsymbol{H1}$ ensures that dynamics are well behaved within $\mathcal{M}_i$:

Assumption. $\boldsymbol{H1}$*: For each $i = 1, \ldots, M$, there exists a compact set of controls $A_i \subset \mathbb{R}^m$, a continuous map $f_i : \mathcal{M}_i \times A_i \to \mathbb{R}^N$, and a payoff function ℓ_i, with the following properties:*

(a) At each $x \in \mathcal{M}_i$, all vectors $f_i(x, a)$, $a \in A_i$ are tangent to $\mathcal{M}_i$;

(b) $|f_i(x, a) - f_i(z, a)| \leq K_i |x - z|$, for some $K_i \in [0, \infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$;

(c) Each payoff function ℓ_i is non-positive and $|\ell_i(x, a) - \ell_i(z, a)| \leq L_i |x - z|$, for some $L_i \in [0, \infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$;[1]

[1]We strengthen part (c) to incorporate the Lipschitz continuity condition stated in BH equation (46).

(d) We have $f(x,a) = f_i(x,a)$ and $\ell(x,a) = \ell_i(x,a)$ whenever $x \in \mathcal{M}_i$, $i = 1, \ldots, M$.

The key assumption is (b); namely, that dynamics are Lipschitz continuous when confined to tangent trajectories. This does not restrict how the dynamics change when crossing the boundaries of $\mathcal{M}_i$.

Let $\mathcal{T}_{\mathcal{M}_i}(x)$ denote the cone of trajectories tangent to $\mathcal{M}_i$ at $x \in \mathcal{M}_i$:

$$\mathcal{T}_{M_i}(x) \equiv \left\{ y \in \mathbb{R}^N \middle| \lim_{h \to 0} \frac{\inf_{z \in \mathcal{M}_i} |x + hy - z|}{h} = 0 \right\}.$$

Part (a) of $\boldsymbol{H1}$ is equivalent to $f_i(x,a) \in \mathcal{T}_{\mathcal{M}_i}$ for all $x \in \mathcal{M}_i, a \in A_i$.

For $x \in \mathcal{M}_i$, let $\hat{F}(x) \subset \mathbb{R}^{N+1}$ denote the set of achievable dynamics and payoffs for the set of controls A_i:

$$\hat{F}(x) \equiv \{(\dot{x}, u) | \dot{x} = f_i(x,a), u \leq \ell_i(x,a), a \in A_i\}, \tag{2}$$

where i is such that $x \in \mathcal{M}_i$. To handle discontinuous dynamics, BH use results from differential inclusions. In particular, let $G(x)$ denote an extended set of feasible trajectories and payoffs:

$$G(x) \equiv \cap_{\epsilon > 0} \overline{co} \left\{ (\dot{x}, u) \in \hat{F}(x') \middle| |x - x'| < \epsilon \right\}, \tag{3}$$

where $\overline{co}S$ denotes the closed convex hull of a set S.

The next key assumption is that $G(x)$ does not contain additional trajectory-payoff pairs when restricted to tangent trajectories:

Assumption. $\boldsymbol{H2}$: *For every $x \in \mathbb{R}^N$, we have*

$$\hat{F}(x) = \left\{ (\dot{x}, u) \in G(x) | \dot{x} \in \mathcal{T}_{\mathcal{M}_i} \right\}. \tag{4}$$

BH define the Hamiltonian using $G(x)$ as the relevant choice set:

$$H(x,p) \equiv \sup_{(\dot{x},u) \in G(x)} \{u + p\dot{x}\}. \tag{5}$$

The corresponding HJB is

$$\beta w(x) = H(x, Dw(x)), \tag{6}$$

where D is the differential operator. BH define the following concepts:

Definition 1. *A continuous function w is a **lower solution** of (6) if the following holds: If $w - \varphi$ has a local maximum at x for some continuously differential φ, then*

$$\beta w(x) - H(x, D\varphi(x)) \leq 0. \tag{7}$$

Definition 2. *A continuous function w is an **upper solution** of (6) if the following holds: If $x \in \mathcal{M}_i$, and the restriction of $w - \varphi$ to $\mathcal{M}_i$ has a local minimum at x for some continuously differential φ,* *then*

$$\beta w(x) - \sup_{(\dot{x},u)\in G(x), \dot{x}\in T_{\mathcal{M}_i}} \{u + D\varphi \dot{x}\} \geq 0. \tag{8}$$

Definition 3. *A continuous function w, which is both an upper and a lower solution of (6), is a* ***viscosity solution****.*

Note that the second definition differs from the first by restricting attention to $\mathcal{M}_i$ when describing the properties of $w-\varphi$, which relaxes the set of φ that satisfies the condition. However, the trajectories in the Hamiltonian are restricted to lie in the tangent set.[2] The added properties are the core distinction between the definition of viscosity solution used here versus the standard definition.[3]

With these definitions in hand, we summarize the core results of BH:

(i) (BH Theorem 1) If $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold, and there exists at least one trajectory with finite value, then the maximization problem admits an optimal solution.

(ii) (BH Proposition 1) Let assumptions $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold. If the value function W is continuous, then it is a viscosity solution of (6).

(iii) (BH Corollary 1) Let assumptions $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold. If the value function W is bounded and Lipschitz continuous, then W is the unique non-positive viscosity solution to (6).[4]

1.2 The Planner's Problem

To map problem (3) into BH, we make a few modifications and consider a generalized problem. First, we let the planner randomize when the government is indifferent to default or not. This helps to convexify the choice set. In particular, let $\pi(t) \in [0,1]$ be an additional choice, where $\pi(t)$ is the probability the government defaults when $\overline{V}$ arises and the current value is $\overline{V}$. It will always be efficient to set $\pi(t) = 0$ when $v(t) = \overline{V}$, and so this does not alter the solution to the planner's problem. We denote the set of possible paths, $\boldsymbol{\pi} = \{\pi(t) \in [0,1]\}_{t\geq 0}$, by $\boldsymbol{\Pi}$. The controls are $\boldsymbol{\alpha} = (\boldsymbol{c}, \boldsymbol{\pi}) \in \mathcal{A} \equiv \boldsymbol{C} \times \boldsymbol{\Pi}$.

Recall that in (3) the objective is discounted by the probability of repayment, $e^{-\lambda \int_0^t \mathbb{1}_{[v(s)<\overline{V}]} ds}$. Let us define $\Gamma(t)$ as follows:

$$\Gamma(t) \equiv \Gamma(0) e^{-\lambda \int_0^t (\pi(s)\mathbb{1}_{[v(s)=\overline{V}]} + \mathbb{1}_{[v(s)<\overline{V}]} ds}$$

for some $\Gamma(0) \in (0,1]$. Note that $\Gamma(t)/\Gamma(0)$ is the discount factor in the original problem with $\pi = 0$. By adding $\Gamma(t)$ as an additional state variable, we will be able to keep track of the probability of survival in our recursive formulation.

[2]The fact that trajectories are restricted to T_{M_i} in the definition of an upper solution was unintentionally omitted in Bressan and Hong (2007) but is corrected in Bressan (2013).

[3]Note that we place the restriction on the upper solution while the original BH place it on the lower solution as we consider a maximization problem.

[4]BH state a weaker continuity condition than Lipschitz continuity (BH $\boldsymbol{H3}$) that is not necessary given our environment.

Let $X = \mathbb{V} \times (0,1]$ denote the state space for $x = (v, \Gamma)$. Let $f(x, \alpha) = (\dot{v}, \dot{\Gamma})$ given the control $\alpha = (c, \pi)$:

$$f(x,\alpha) = \begin{cases} \dot{v} & = -c + \rho v - \mathbb{1}_{[v<\overline{V}]}\lambda \left[\overline{V} - v\right] \\ \dot{\Gamma} & = -\lambda \left[\pi \mathbb{1}_{[v=\overline{V}]} + \mathbb{1}_{[v<\overline{V}]}\right]\Gamma. \end{cases} \tag{9}$$

The flow value must be non-positive. We therefore subtract the constant $(y - \underline{C})/r$ from the value. To convert this into a flow payoff, let

$$\ell(x, a) \equiv \Gamma(y - c) - (y - \underline{C}) \leq 0,$$

where the inequality uses $y > \underline{C}$ and $\Gamma \leq 1$. Note that ℓ is Lipschitz continuous in x.

Hence, we consider the following problem, where $x(t) \equiv (v(t), \Gamma(t))$:

$$\tilde{P}(v,\Gamma) = \sup_{\alpha \in \mathcal{A}} \int_0^\infty e^{-rt}\ell(x(t), \alpha(t))\mathrm{d}t \tag{10}$$
$$\text{subject to } \begin{cases} (v(0), \Gamma(0)) & = (v, \Gamma) \\ (\dot{v}(t), \dot{\Gamma}(t)) & = f(x(t), \alpha(t)). \end{cases}$$

We then have a one-to-one mapping between $\tilde{P}$ and $P^\star$:

$$\tilde{P}(v,\Gamma) = \Gamma P^\star(v) - (y - \underline{C})/r. \tag{11}$$

As $P^\star$ is bounded and $\Gamma \in (0, 1]$, $\tilde{P}$ is bounded. Similarly, $\tilde{P}$ is Lipschitz continuous in the state vector (v, Γ).

We now verify the conditions of BH. Define five regions of the state space:

$$\begin{aligned} \mathcal{M}_1 &\equiv \{\underline{V}\} \times (0,1] \\ \mathcal{M}_2 &\equiv (\underline{V}, \overline{V}) \times (0,1] \\ \mathcal{M}_3 &\equiv \{\overline{V}\} \times (0,1] \\ \mathcal{M}_4 &\equiv (\overline{V}, V_{max}) \times (0,1] \\ \mathcal{M}_5 &\equiv \{V_{max}\} \times (0,1]. \end{aligned}$$

Let A_i denote the controls that produce trajectories that are tangent to $\mathcal{M}_i$:[5]

$$\begin{aligned} A_i &\equiv \{(c,\pi) | c \in [\underline{C}, \overline{C}], \pi \in [0,1], \dot{x} \in \mathcal{T}_{M_i}\} \\ &= \begin{cases} \{\rho \underline{V} - \lambda(\overline{V} - \underline{V})\} \times [0,1] \text{ if } i = 1 \\ \{\rho \overline{V}\} \times [0,1] \text{ if } i = 3 \\ \{\rho V_{max}\} \times [0,1] \text{ if } i = 5 \\ [\underline{C}, \overline{C}] \times [0,1] \text{ otherwise.} \end{cases} \end{aligned} \tag{12}$$

[5]For $i = 1, 3, 5$, the tangent trajectories set $\dot{v} = 0$. Otherwise, they are the full set of trajectories.

Within each $\mathcal{M}_i$, the dynamics f are Lipschitz continuous in x for all $a \in A_i$. It is straightforward to verify that we satisfy Assumption $\boldsymbol{H1}$.

Let us now verify Assumption $\boldsymbol{H2}$. There two cases:

Case 1: $i \in \{2, 4\}$. In this case, $G(x) = \hat{F}(x)$, and BH Assumption $\boldsymbol{H2}$ is straightforward to verify.

Case 2: $i \in \{1, 3, 5\}$. We show the $i = 3$ case (as the others are similar). We have[6]

$$\hat{F}(x) = \Big\{(\dot{x}, u) | \dot{v} = 0, \dot{\Gamma} = -\pi\lambda\Gamma, u \le \ell(x, (\rho\overline{V}, \pi)), \pi \in [0, 1]\Big\} \tag{13}$$

$$= \Big\{(\dot{x}, u) | \dot{v} = 0, \dot{\Gamma} \in [-\lambda\Gamma, 0], u \le \Gamma(y - \rho\overline{V}) - (y - \underline{C})\Big\}$$

$$= \{0\} \times [-\lambda\Gamma, 0] \times (-\infty, \Gamma(y - \rho\overline{V}) - (y - \underline{C})]. \tag{14}$$

Let $x' = (v', \Gamma')$ be in the neighborhood of $x = (\overline{V}, \Gamma)$. We have

$$\begin{aligned} \hat{F}(x') = \Big\{(\dot{x}, u) \Big| \\ & \dot{v} = -c + \rho v' - \lambda \mathbb{1}_{\{v' < \overline{V}\}}(\overline{V} - v'), \\ & \dot{\Gamma} \in [-\lambda \mathbb{1}_{\{v' < \overline{V}\}}\Gamma, 0], \\ & u \le \Gamma(y - c) - (y - \underline{C}), c \in [\underline{C}, \overline{C}]\Big\}. \end{aligned}$$

We have that

$$\begin{aligned} \cup_{|x'-x| \le \epsilon} \hat{F}(x') \subseteq R(x, \epsilon) \equiv \Big\{ & \dot{v} = -c + \theta, \\ & \dot{\Gamma} = [-\lambda(\Gamma + \epsilon), 0], \\ & u \le (\Gamma + \epsilon - 1)y - (\Gamma - \epsilon)c + \underline{c}, \\ & \theta \in [\rho(\overline{V} - \epsilon) - \lambda\epsilon, \rho(\overline{V} + \epsilon)] \\ & c \in [\underline{C}, \overline{C}]\Big\}. \end{aligned}$$

Note that $R(x, \epsilon)$ is convex and $G(x) = \cap_{\epsilon > 0} R(x, \epsilon)$. Also note that

$$\hat{F}(x) = \{(\dot{x}, u) \in G(x) | \dot{x} \in \mathcal{T}_{\mathcal{M}_3}\},$$

where the equivalence uses the definitions of G, $\hat{F}$, and the tangent trajectories $\mathcal{T}_{\mathcal{M}_3}$. This verifies BH $\boldsymbol{H2}$ for $\mathcal{M}_3$.

Similar steps hold for $i = 1$ and 5, verifying Assumption $\boldsymbol{H2}$ for all domains.[7]

As noted above, $\tilde{P}$ is bounded and Lipschitz continuous. Hence, by BH Corollary 1, it is the

[6]Note this is the only case where the choice of π is relevant.

[7]For $v = \underline{V}$, we extend the dynamics to both sides of $\underline{V}$ by setting $\dot{v} = -c + \rho v - \lambda(\overline{V} - v)$ in the neighborhood $v < \underline{V}$ and ℓ arbitrarily low. Thus, the dynamics are continuous at $x = (\underline{V}, \Gamma)$. Similarly for $v = V_{max}$, we set $\dot{v} = -c + \rho v$.

unique viscosity solution with such properties for the HJB:

$$r\tilde{P}(v,\Gamma) = H((v,\Gamma),(\tilde{P}_V,\tilde{P}_\Gamma)) \equiv \sup_{(c,\pi)\in[\underline{C},\overline{C}]\times[0,1]} \left\{\Gamma(y-c) - (y-\underline{C}) + \tilde{P}_v\dot{v} + \tilde{P}_\Gamma\dot{\Gamma}\right\}, \tag{15}$$

where $\dot{v}$ and $\dot{\Gamma}$ obey equation (9). Here, we have used the fact that $G(x)$ contains the full set of trajectories generated by $c \in [\underline{C},\overline{C}]$ and $\pi \in [0,1]$. Note that it is optimal to set π to 0, and thus we can ignore this choice in the Hamiltonian in what follows. We shall use the fact that H is convex in $\tilde{P}_v$.

1.3 Proof of Proposition B.1

Proof. Suppose that $p(v)$ satisfies the conditions in the proposition. We shall show that $\tilde{p}(v,\Gamma) \equiv \Gamma p(v) - (y-\underline{C})/r$ is a viscosity solution of (15). $\tilde{p}$ is differentiable in Γ at all points, and in v at points where $p(v)$ is differentiable. We now check the conditions for a viscosity solution. We proceed by checking on each domain $\mathcal{M}_i$.

(i) $(v,\Gamma) \in \mathcal{M}_2 \cup \mathcal{M}_4$

As p is differentiable on this part of the domain, by condition (i) of the proposition, we have

$$\begin{aligned} rp(v) &= \sup_{c\in[\underline{C},\overline{C}]} \left\{y - c + p'(v)\dot{v} + \mathbb{1}_{[v<\overline{V}]}p(v)\right\} \\ &= \sup_{c\in[\underline{C},\overline{C}]} \left\{y - c + \Gamma^{-1}\tilde{p}_v\dot{v} + \Gamma^{-1}\tilde{p}_\Gamma\dot{\Gamma}\right\}, \end{aligned}$$

where the second line uses $\tilde{p}_v = \Gamma p'(v)$ and $\tilde{p}_\Gamma\dot{\Gamma}/\Gamma = -\lambda\mathbb{1}_{[v<\overline{V}]}p$. Multiplying through by $\Gamma \in (0,1]$ and subtracting $(y-\underline{C})/r$ from both sides yields

$$\begin{aligned} r\tilde{p}(v) = r(\Gamma p(v) - (y-\underline{C})) &= \sup_{c\in[\underline{C},\overline{C}]} \left\{\Gamma(y-c) - (y-\underline{C})/r + \tilde{p}_v\dot{v} + \tilde{p}_\Gamma\dot{\Gamma}\right\} \\ &= H((v,\Gamma),(\tilde{p}_v,\tilde{p}_\Gamma)). \end{aligned}$$

Hence, $\tilde{p}$ satisfies (15).

Now consider a point of non-differentiability $\tilde{v}$. As $(v,\Gamma) \notin \mathcal{M}_3$, $\tilde{v} \neq \overline{V}$, and hence condition (iii) of the proposition is applicable. Condition (iii) states that $p^-_{\tilde{v}} \equiv \lim_{v\uparrow\tilde{v}} p'(v) < \lim_{v\downarrow\tilde{v}} p'(v) \equiv p^+_{\tilde{v}}$). Hence, there is a convex kink. In this case, the lower solution does not impose additional conditions, leaving the conditions for an upper solution to be verified. Suppose φ is differentiable and $\tilde{p} - \varphi$ has a local minimum at $(\tilde{v},\Gamma)$. Then $\varphi_v \in [p^-_{\tilde{v}}, p^+_{\tilde{v}}]$. As $\tilde{p}$ is differentiable in Γ, we have $\varphi_\Gamma = \tilde{p}_\Gamma$. Note that

$$r\tilde{p}(\tilde{v}) = H((\tilde{v},\Gamma),(p^-_{\tilde{v}},\tilde{p}_\Gamma)) = H((\tilde{v},\Gamma),(p^+_{\tilde{v}},\tilde{p}_\Gamma)), \tag{16}$$

as the HJB holds with equality at points of differentiability in the neighborhood of $\tilde{v}$, and using the continuity of H.

Note that there exists $\alpha \in [0,1]$ such that $\varphi_v = \alpha p^+_{\tilde{v}} + (1-\alpha)p^-_{\tilde{v}}$. The convexity of H with respect to φ_v implies that

$$\begin{aligned} H((\tilde{v},\Gamma),(\varphi_v,\varphi_\Gamma)) &= H((\tilde{v},\Gamma),(\alpha p^+_{\tilde{v}} + (1-\alpha)p^-_{\tilde{v}},\varphi_\Gamma)) \\ &\leq \alpha H((\tilde{v},\Gamma),(p^+_{\tilde{v}},\varphi_\Gamma)) + (1-\alpha)H((\tilde{v},\Gamma),(p^-_{\tilde{v}},\varphi_\Gamma)) \\ &= r\tilde{p}(\tilde{v}), \end{aligned}$$

where the last equality uses (16) and $\varphi_\Gamma = \tilde{p}_\Gamma$. Hence, $\tilde{p}(\tilde{v})$ satisfies the conditions of an upper solution.

(ii) $(v,\Gamma) \in \mathcal{M}_3 = \{\overline{V}\} \times (0,1]$

Turning to $v = \overline{V}$, we redefine $p^+_v \equiv \lim_{v \downarrow \overline{V}} p'(v)$ and $p^-_v \equiv \lim_{v \uparrow \overline{V}} p'(v)$. Given the continuity of p and the fact that it satisfies the HJB in the neighborhood of $\overline{V}$ with equality, we have

$$\begin{aligned} rp(\overline{V}) &= \sup_{c \in [\underline{C},\overline{C}]} \{y - c + p^+_v \dot{v}\} \\ &= \sup_{c \in [\underline{C},\overline{C}]} \{y - c + p^-_v \dot{v} - \lambda p(\overline{V})\}, \end{aligned} \tag{17}$$

where $\dot{v} = -c + \rho\overline{V}$. As setting $c = \rho\overline{V}$ is always feasible, this implies $rp(\overline{V}) \geq (y - \rho\overline{V}) \geq 0$.

To verify that $\tilde{p}$ is a viscosity solution to (5), note that if $\tilde{p}$ is differentiable, then it satisfies the HJB with equality by condition (i) of the proposition.

If it is not differentiable, we consider convex and concave kinks in turn.

Suppose that $p^-_v < p^+_v$. Then the conditions for a lower solution do not impose any restrictions. For an upper solution, consider a φ such that $\tilde{p} - \varphi$ has a local minimum at $(\overline{V},\Gamma)$. Again, $\varphi_\Gamma = \tilde{p}_\Gamma = p(\overline{V})$. Recall that for an upper solution, we need only consider trajectories that are in $\mathcal{T}_{\mathcal{M}_3}$, that is, $\dot{v} = 0$ and thus $c = \rho\overline{V}$. Hence:

$$\begin{aligned} r\tilde{p}(\overline{V}) &= r\Gamma p(\overline{V}) - (y - \underline{C}) \\ &\geq \Gamma(y - \rho\overline{V}) - (y - \underline{C}) \\ &= \sup_{c=\rho\overline{V}} \left\{ \Gamma(y-c) - (y - \underline{C}) + \varphi_v \underbrace{\left(-c + \rho\overline{V}\right)}_{\dot{v}} \right\} \\ &= \sup_{c=\rho\overline{V}, \pi\in[0,1]} \left\{ \Gamma(y-c) - (y - \underline{C}) + \varphi_v \underbrace{\left(-c + \rho\overline{V}\right)}_{\dot{v}} - \underbrace{p(\overline{V}) \times \pi\lambda\Gamma}_{\varphi_\Gamma \times \dot{\Gamma}} \right\}, \end{aligned}$$

where the last equality uses $p(\overline{V}) \geq 0$. Note that final term is the Hamiltonian maximized along tangent trajectories in $\mathcal{T}_{\mathcal{M}_3}$. Thus, the conditions of an upper solution are satisfied.

For the lower solution, we must consider the case in which $\tilde{p} - \varphi$ has a local maximum at

$(\overline{V},\Gamma)$. This requires $p_v^- \geq p_v^+$ and $\varphi_v \in [p_v^+, p_v^-]$. Again, as $\tilde{p}$ is differentiable with respect to Γ, we have $\varphi_\Gamma = \tilde{p}_\Gamma = p(\overline{V})$.

If $p_v^+ \leq -1$, then condition (ii) in the proposition implies that

$$\begin{aligned} r\tilde{p}(\overline{V}) &= \Gamma(y-\rho\overline{V}) - (y-\overline{C}) \\ &\leq \sup_{c\in[\underline{C},\overline{C}],\pi\in[0,1]} \left\{ \Gamma(y-c) - (y-\overline{C}) + \varphi_v \underbrace{(-c+\rho\overline{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi\lambda\Gamma)}_{\dot{\Gamma}} \right\} \\ &= H((\overline{V},\Gamma),(\varphi_v,\varphi_\Gamma)), \end{aligned}$$

where the second to the last line follows from $\varphi_\Gamma = p(\overline{V}) \geq 0$. Hence, $\tilde{p}(\overline{V}) = \Gamma p(\overline{V}) - (y-\underline{C})/r$ satisfies the condition for a lower solution when $p_v^+ \leq -1$.

Alternatively, if $p_v^+ > -1$, then

$$\begin{aligned} rp(\overline{V}) &= \sup_{c\in[\underline{C},\overline{C}]} \{y - c + p_v^+(-c+\rho\overline{V})\} \\ &= y - \underline{C} + p_v^+(\rho\overline{V} - \underline{C}) \\ &\leq y - \underline{C} + \varphi_v(\rho\overline{V} - \underline{C}), \end{aligned}$$

for $\varphi_v \geq p_v^+$ as $\rho\overline{V} > \underline{C}$. Hence,

$$r\tilde{p}(\overline{V}) \leq \sup_{c\in[\underline{C},\overline{C}],\pi\in[0,1]} \left\{ \Gamma(y-c) - (y-\overline{C}) + \varphi_v \underbrace{(-c+\rho\overline{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi\lambda\Gamma)}_{\dot{\Gamma}} \right\}$$

for $\varphi_v \in [p_v^+, p_v^-]$ and $\varphi_\Gamma = p(\overline{V})$, satisfying the condition for a lower solution.

(iii) $(v,\Gamma) \in \mathcal{M}_1 = \{\underline{V}\} \times (0,1]$

For $v = \underline{V}$, the condition for $\tilde{p}$ to be an upper solution is

$$r\tilde{p}(\underline{V},\Gamma) \geq \Gamma\left(y - \rho\underline{V} + \lambda(\overline{V}-\underline{V})\right) - (y-\underline{C}) - \lambda p(\underline{V})\Gamma,$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v} = 0$. As $\tilde{p}$ satisfies the HJB with equality in the neighborhood of $\underline{V}$, we have

$$\begin{aligned} p(\underline{V},\Gamma) = \lim_{v\downarrow\underline{V}} r\tilde{p}(v,\Gamma) &= \lim_{v\downarrow\underline{V}} H((v,\Gamma),(\Gamma p'(v),p(v))) \\ &\geq \lim_{v\downarrow\underline{V}} \left\{\Gamma\left(y-\rho v+\lambda(\overline{V}-v)\right) - (y-\underline{C}) - \lambda p(v)\Gamma, \right\} \\ &= \Gamma\left(y-\rho\underline{V}+\lambda(\overline{V}-\underline{V})\right) - (y-\underline{C}) - \lambda p(\underline{V})\Gamma. \end{aligned}$$

Hence, $\tilde{p}$ is an upper solution.

Turning to the lower solution, suppose $\tilde{p} - \varphi$ has a local maximum at $(\underline{V}, \Gamma)$. As $\underline{V}$ is at the boundary of the state space, this implies $\varphi_v \geq \tilde{p}_v(\underline{V}, \Gamma)$ and $\varphi_\Gamma = p(\underline{V})$. A lower solution requires

$$r\tilde{p}(\underline{V}, \Gamma) \leq H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)).$$

Suppose $p'(\underline{V}) < -1$. Then, condition (iv) of the proposition implies

$$\begin{aligned}
r\tilde{p}(\underline{V}, \Gamma) &= r\Gamma p(\underline{V}) - (y - \underline{C}) \\
&= r\Gamma \left(\frac{y - \rho \underline{V} + \lambda(\overline{V} - \underline{V})}{r + \lambda} \right) - (y - \underline{C}) \\
&= \left(1 - \frac{\lambda}{r + \lambda}\right) \Gamma \left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) - (y - \underline{C}) \\
&\leq \sup_{c \in [\underline{C}, \overline{C}]} \left\{ \Gamma(y - c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho \underline{V} - \lambda(\overline{V} - \underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda \Gamma)}_{\dot{\Gamma}} \right\} \\
&= H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)),
\end{aligned}$$

where the inequality uses

$$\begin{aligned}
-\varphi_\Gamma \lambda \Gamma &= -p(\underline{V}) \lambda \Gamma \\
&= -\left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) \frac{\lambda}{r + \lambda} \Gamma.
\end{aligned}$$

This verifies that $\tilde{p}$ is a lower solution if $p'(\underline{V}) < -1$.

Turning to $p'(\underline{V}) \geq -1$,

$$\begin{aligned}
&H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)) = \\
&\quad = \sup_{c \in [\underline{C}, \overline{C}]} \left\{ \Gamma(y - c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho \underline{V} - \lambda(\overline{V} - \underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda \Gamma)}_{\dot{\Gamma}} \right\} \\
&\quad = \Gamma(y - \underline{C}) - (y - \overline{C}) + \varphi_v \underbrace{(-\underline{C} + \rho \underline{V} - \lambda(\overline{V} - \underline{V}))}_{>0} + \varphi_\Gamma(-\lambda \Gamma) \\
&\quad \geq \Gamma(y - \underline{C}) - (y - \overline{C}) + \tilde{p}_v(\underline{V}, \Gamma)(-\underline{C} + \rho \underline{V} - \lambda(\overline{V} - \underline{V})) + \varphi_\Gamma(-\lambda \Gamma) \\
&\quad = H((\underline{V}, \Gamma), (\tilde{p}_v, \tilde{p}_\Gamma)) = r\tilde{p}(\underline{V}, \Gamma),
\end{aligned}$$

where the second equality uses the fact that $\underline{C}$ is optimal when $\varphi_v \geq \Gamma p'(v) \geq -\Gamma$; the inequality uses the fact that $\varphi_v \geq \tilde{p}_v$ and the term multiplying φ_v is positive; and the last line uses the continuity of the Hamiltonian and the value function, and that $\underline{C}$ is optimal given $p'(\underline{V}) \geq -1$. This verifies that $\tilde{p}$ is a lower solution if $p'(\underline{V}) \geq -1$.

(iv) $(v, \Gamma) \in \mathcal{M}_5 = \{V_{max}\} \times (0, 1]$

For $v = V_{max}$, the condition for $\tilde{p}$ to be an upper solution is

$$r\tilde{p}(V_{max}, \Gamma) \geq \Gamma\,(y - \rho V_{max}) - (y - \underline{C}),$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v} = 0$. As $\tilde{p}$ satisfies the HJB with equality in the neighborhood of V_{max}, we have

$$\begin{aligned} r\tilde{p}(V_{max}, \Gamma) = \lim_{v \uparrow V_{max}} r\tilde{p}(v, \Gamma) &= \lim_{v \uparrow V_{max}} H((v, \Gamma), (\Gamma p'(v), p(v))) \\ &\geq \lim_{v \uparrow V_{max}} \left\{ \Gamma\,(y - \rho v) - (y - \underline{C}) \right\} \\ &= \Gamma\,(y - \rho V_{max}) - (y - \underline{C}). \end{aligned}$$

Hence, $\tilde{p}$ is an upper solution.

For the lower solution, suppose $\tilde{p} - \varphi$ has a local maximum at (V_{max}, Γ). This implies $\varphi_v \leq \tilde{p}_v = \Gamma p'(V_{max})$ and $\varphi_\Gamma = p(V_{max})$. The condition for a lower solution is

$$\begin{aligned} r\tilde{p}(V_{max}, \Gamma) &\leq \sup_{c \in [\underline{C}, \overline{C}]} \left\{ \Gamma(y - c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho V_{max})}_{\dot{v}} \right\} \\ &= H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)). \end{aligned}$$

By condition (v) of the proposition, we have $p'(V_{max}) \leq -1$, implying that $\varphi_v \leq -\Gamma$. Hence, $c = \overline{C}$ achieves the optimum in $H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma))$. That is,

$$\begin{aligned} H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= \Gamma(y - \overline{C}) - (y - \overline{C}) + \varphi_v \underbrace{(-\overline{C} + \rho V_{max})}_{\dot{v}} \\ &\geq \Gamma(y - \overline{C}) - (y - \overline{C}) + \tilde{p}_v(-\overline{C} + \rho V_{max}) \\ &= r\tilde{p}(V_{max}, \Gamma), \end{aligned}$$

where the inequality uses $\varphi_v \geq \tilde{p}_v$ and $\overline{C} \geq \rho V_{max}$; and the final line uses continuity of H and $\tilde{p}$. Hence, $\tilde{p}$ is a lower solution at (V_{max}, Γ).

We have shown that $\tilde{p}$ implied by a p satisfying the conditions of the proposition is a viscosity solution of the planner's problem. □

1.4 The Competitive Equilibrium

This section maps the government's problem into the BH framework.

Let us first define the following operator T that takes as an input a candidate value function, $\tilde{V}(b)$, assumed to be bounded and Lipschitz continuous, and a debt dynamics function $f(b, c)$ that embeds the price function, $q(b)$:

$$T\tilde{V}(b) = \int_0^\infty e^{-(r+\lambda)t} \left(c(t) + \lambda D(b(t)|\tilde{V})\right) \tag{18}$$

$$\begin{aligned}&\text{subject to:}\\ &\quad \dot{b}(t) = f(b(t), c(t))\\ &\quad b(0) = b,\end{aligned}$$

where

$$D(b|\tilde{V}) \equiv \mathbb{1}_{[\tilde{V}(b)<\overline{V}]}(\overline{V} - \tilde{V}(b)).$$

The government's equilibrium value function is a fixed point of this operator. We shall map the right-hand side problem into the BH framework and use recursive techniques to solve the optimization. Toward this goal, let

$$\ell(b,c) \equiv c + \lambda D(b|\tilde{V}).$$

Note that $\ell(b,c)$ so defined is Lipschitz continuous and bounded. To be consistent with BH, we also need a non-positive ℓ. This can be achieved by subtracting the maximum value of ℓ. Rather than carrying this notation through, we proceed with the objects defined above, recognizing that all flow utilities and values can be appropriately translated (as we did explicitly in the planning problem).

Turning to the dynamics, $f(b,c)$, suppose the government faces a closed, convex domain $\boldsymbol{B}$ and an equilibrium price schedule $q : \boldsymbol{B} \to [\underline{q}, 1]$ that is differentiable almost everywhere with $|q'(b)| < M < \infty$.

Let $b_0 \equiv -\overline{a}$; $b_1, ..., b_N$ denote the N points of non-continuity in q; and $b_{N+1} \equiv \overline{b}$. We consider equilibria in which $\limsup_{b \to b_n} q(b) = q(b_n)$, as our tie-breaking rule is that the government chooses the action that maximizes the price when indifferent.

To define the domains, let $\mathcal{M}_n \equiv (b_{n-1}, b_n)$, $n = 1, ..., N+1$, be the open sets on which q is differentiable. Let $\mathcal{M}_{N+1+n} \equiv \{b_n\}$, $n = 1, ..., N$ be the isolated points of non-differentiability. Finally, we have the endpoints of the domain: $\{-\overline{a}\}$ and $\{\overline{b}\}$.

In the neighborhood of a discontinuity, we rule out repurchases at the "low price" (see footnotes 28 and 31). We do this while ensuring the continuity of dynamics. Specifically, let $\Delta > 0$ be arbitrarily small; and in particular, $\Delta < \inf_n |b_n - b_{n-1}|/2$. Define $\alpha(b) \equiv \min\{|b - b_n|/\Delta, 1\}$, where b_n is the closest point of non-differentiability to b. Note that $\alpha(b) \in [0,1]$, and equals one if $|b - b_n| \geq \Delta$. Debt dynamics are given by

$$f(b,c) = \begin{cases} \frac{c-y+(r+\delta)b}{q(b)} - \delta b & \text{if } c \geq y - (r+\delta)b \\ \frac{c-y+(r+\delta)b}{\alpha(b)q(b)+(1-\alpha(b))q(b_n)} - \delta b & \text{if } c < y - (r+\delta)b. \end{cases} \tag{19}$$

Note that $f(b,c)$ is Lipschitz continuous in b and c *within the domains* $\mathcal{M}_n$.

On the open sets $\mathcal{M}_n$, $n = 1, ..., N+1$, any $c \in A_n \equiv [\underline{C}, \overline{C}]$ results in a tangent trajectory. For $n > N+1$, $c \in A_n \equiv y - [r + \delta(1 - q(b_n))]b_n$ is the singleton set that generates a tangent trajectory to the isolated point $\mathcal{M}_n$. Hence, BH assumption $\boldsymbol{H1}$ is satisfied.

Following BH, define

$$\hat{F}(b) \equiv \left\{(\dot{b}, u) \middle| \dot{b} = f(b, c), u \leq \ell(b, c), c \in A_n\right\}. \tag{20}$$

If $b = b_n$ for some n, we have

$$\hat{F}(b_n) = \{0\} \times \{u \leq \ell(b, y - [r + \delta(1 - q(b_n))]b_n)\}. \tag{21}$$

Otherwise,

$$\hat{F}(b) = \left\{(\dot{b}, u) \middle| \dot{b} \in [f(b, \underline{C}), f(b, \overline{C})], u \leq \ell(b, q(b)(\dot{b} + \delta b) + y - (r + \delta)b)\right\}. \tag{22}$$

Finally, define

$$G(b) \equiv \cap_{\varepsilon > 0} \overline{\text{co}} \left\{(\dot{b}, u) \in \hat{F}(b') \text{ such that } |b' - b| < \varepsilon\right\}. \tag{23}$$

To characterize this set, if $b \neq b_n$, then $G(b) = \hat{F}(b)$ as f is continuous within the open set $\mathcal{M}_n$, $n = 1, \dots, N + 1$, and the tangent trajectories are generated by $c \in [\underline{C}, \overline{C}]$. For $b = b_n$ for some n, we have

$$G(b_n) = \left\{(\dot{b}, u) | \dot{b} = f(b, c), u \leq \ell(b, c), c \in [\underline{C}, \overline{C}]\right\}.$$

For this case, restricting attention to $c = y - [r + \delta(1 - q(b_n))]b_n$ yields $\hat{F}(b_n)$. Hence BH assumption $\boldsymbol{H2}$ is satisfied.

We use the assumption regarding repurchases around a point of discontinuity in q to rule out the following. Suppose that the following trajectory was feasible: $\dot{b} < -\delta b$ and $c = \liminf_{b \to b_n} q(b_n)(\dot{b} - \delta b) - (r + \delta)b + y > q(b_n)(\dot{b} - \delta b) - (r + \delta)b + y$. Then, in the convexification generating $G(b_n)$, a trajectory featuring $\dot{b} = 0$ and $c > y - [r + \delta(1 - q(b_n))]b$ would appear. This new trajectory would be generated by locating two trajectories featuring $\dot{b} < -\delta b$ and $\dot{b} > -\delta b$, such that their convex combination leads to $\dot{b} = 0$. Because for the trajectory with $\dot{b} > -\delta b$ we have that $c = \overline{C}$, the associated convex combination of the consumptions of these two trajectories would then be strictly greater than the stationary consumption in $\hat{F}(b_n)$, violating $\boldsymbol{H2}$.

BH Proposition 1 and Corollary 1 then imply that the solution to $T\tilde{V}$ is the unique bounded, Lipschitz continuous viscosity solution to

$$\rho(T\tilde{V})(b) = \sup_{c \in [\underline{C}, \overline{C}]} \left\{c + \lambda D(b|\tilde{V}) + (T\tilde{V})'(b) f(b, c)\right\}.$$

As V is a fixed point of the operator, the government's value V is a viscosity solution to

$$\rho V(b) = H(b, V'(b)) \equiv \sup_{c \in [\underline{C}, \overline{C}]} \left\{c + \lambda \mathbb{1}_{[V(b) < \overline{V}]}(\overline{V} - V(b)) + V'(b) f(b, c)\right\}, \tag{24}$$

where the term $\lambda \mathbb{1}_{[V(b) < \overline{V}]}(\overline{V} - V(b))$ is taken as a given function of b in verifying the viscosity conditions.

1.5 Proof of Proposition B.2

Proof. We need to verify that if v satisfies the conditions of the proposition, it also satisfies the conditions for a viscosity solution. The proof and details parallel that of the proof for Proposition B.1, and we omit some of the identical steps.

Lower solution conditions. In regard to the conditions for a lower solution, condition (i) in the proposition ensures these are met wherever v is differentiable on the interior of $\boldsymbol{B}$. At the boundaries, $-\overline{a}$ and $\overline{b}$, conditions (iv) and (v) of the proposition state that v equals the stationary value. Hence, $\rho v(b) \leq H(b, \varphi'(b))$, $b \in \{-\overline{a}, \overline{b}\}$, for any $\varphi'(b)$, as $\dot{b} = 0$ is always feasible.

For a non-differentiability at $\underline{b}$, the same argument as for $P(\overline{V})$ in the proof of Proposition B.1 applies. That is, if v has a concave kink, then condition (ii) imposes that value must be the stationary value, which is (weakly) less than $H(b, \varphi'(b))$ for any $\varphi'(b)$. For a convex kink, the lower solution does not impose any restrictions.

At all other points of non-differentiability, condition (iii) states that v has a convex kink, and therefore $v - \varphi$ cannot have a local maximum for a smooth function φ. Thus, the lower solution does not impose any restrictions.

Upper solution conditions. For the upper solution, condition (i) of the proposition states that v satisfies the definition of an upper solution wherever it is differentiable. For points of non-differentiability at $\tilde{b} \neq \underline{b}$, first suppose that q is continuous at $\tilde{b}$. Condition (iii) guarantees that v has a convex kink at $\tilde{b}$, and as in the proof of Proposition B.1, then the convexity of $H(b, \varphi'(b))$ in $\varphi'(b)$ ensures the upper solution inequality is satisfied. If q is not continuous at $\tilde{b}$, then the "tangent trajectories" are restricted to $\dot{b} = 0$. Hence, we need to check that v is weakly greater than the stationary value. This is satisfied by a continuity argument that parallels that in the proof of Proposition B.1.

□

References

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